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On sofic-like shifts and flips

(소픽과 유사한 이동공간과 대합사상에 대하여)

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On sofic-like shifts and flips

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Abstract

On sofic-like shifts and flips

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We study generalizations of the notion of sofic shifts and the existence of flips for a shift space. Coded systems and almost sofic shifts generalize sofic shifts in terms of density of periodic points and entropies, respectively. An irreducible sofic shift is both coded and almost sofic. We investigate which properties of irreducible sofic shifts are extended to coded systems and almost sofic shifts.

We also study which shift-flip systems have infinitely many non-conjugate flips. We prove that if an infinite synchronized system has a flip, then there are countably many non-conjugate flips. However, this property can not be extended to coded systems. We construct a coded system which has only two non-conjugate flips.

Key words: sofic shift, flip, coded system, almost sofic shift
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Chapter 1

Introduction

This thesis is concerned with generalizations of the notion of sofic shifts, and the existence of flips.

A sofic shift is a shift space presented by a finite labeled graph [LinM]. It is also a factor of a shift of finite type. Sofic shifts form the smallest collection of shift spaces that includes shifts of finite type and are closed under factor maps [Wei]. Hence a factor of a sofic shift is also sofic. There are other properties of sofic shifts: sofic shifts have a finitary block w, i.e., if uw and wv are allowed blocks then uwv is also allowed [Kit]; and finitely many futures of left-infinite sequences [Kri1, LinM]. A shift space X is almost sofic if its entropy $h(X)$ can be approximated by sofic subshifts of X [LinM]. It is clear that a sofic shift is almost sofic. Furthermore, if a sofic shift X is irreducible, then it has many periodic points so that $h(X)$ can be obtained from the numbers of periodic points:

$$\limsup_{n\to\infty} \frac{1}{n} \log p_n(X) = h(X),$$

where $p_n(X)$ is the number of points of period n ([LinM, Theorem 4.3.6], [Pet]). If a shift space satisfies the above equation, it is said to be periodic saturated [Pet]. It is clear that an almost sofic shift is periodic saturated (Remark 2.2.2).

Our work is to study the above properties in shift spaces presented by countable labeled graphs $\mathcal{G}$. A countable graph consists of a countable vertex set and a countable edge set. A countable labeled graph $\mathcal{G}$ is a countable graph with a labeling of edges by a finite set. In general the set X of all bi-infinite sequences presented by $\mathcal{G}$ is not closed, but it is shift-invariant. Hence the closure $\overline{X}$ of X is a shift space. If $\mathcal{G}$ is irreducible then the shift space $\overline{X}$, called a coded system, generalizes one of the above properties for sofic shifts: a factor of a coded system is coded [BlaH]. A shift space is called a synchronized system if it is an irreducible shift space having a finitary block [BlaH]. This shift space is also coded ([BlaH, Proposition 4.1], Theorem 3.1.4). It is clear that an irreducible sofic shift is synchronized, but the class of synchronized systems is not closed under factor maps [BlaH]. Some other relations between sofic shifts and synchronized systems are developed in [FieF1].

Not every coded system satisfies the other properties. In [FieF1], they modified the Dyck system and obtained a coded systems such that it has uncountably many futures of left-infinite sequences. In Section 5.2 we construct a coded system which is not periodic saturated. Moreover, the coded system is not almost sofic.

Hence we obtain the following: let X be an irreducible shift space.

(1) If X is sofic, then it is synchronized, and coded.

(2) If X is sofic, then it is almost sofic, and periodic saturated.

(3) There is an X such that it is neither almost sofic nor coded. But it is periodic saturated.

(4) There is a coded system which is not almost sofic.

These statements are shown in Chapters 3, 4 and 5. We explain them briefly here. The statement (1) is shown in Section 3.1. Of course, the first implication of (2) hold by definition of almost sofic shift, but it is also true in the sense of [Pet], and its proof is given in Section 3.2. The second implication of (2) follows from Remark 2.2.2. In Chapter 4 we construct a shift space which has the properties stated in (3). The shift space is constructed in [Pet], and it has other properties. The statement (4) is shown in Section 5.2, in fact, we construct a synchronized system which is not periodic saturated. Conversely there is an almost sofic shift which is not coded (Section 5.4).

A homeomorphism $\varphi : (X, \sigma_X) \to (X, \sigma_X)$ is called a reversal for (X, σ_X) if $\varphi\sigma_X = \sigma_X^{-1}\varphi$ [LeePS]. If a positive integer n satisfies the following: $\varphi^n = \mathrm{id}_X$ and $\varphi^k \neq \mathrm{id}_X$ $(1 \leq k \leq n-1)$, then we call n the order of φ. If the order of φ is $2m-1$ for some $m \geq 1$, then we have $\sigma_X^2 = \mathrm{id}_X$ since $\varphi^{2m}\sigma_X = \sigma_X\varphi^{2m}$. Hence we are interested in the case when the order of φ is even. In this thesis we focus on φ with order 2. In this case φ is called a flip for (X, σ_X), and the triple (X, σ_X, φ) is called a shift-flip system [ChoK, KimLP, LeePS]. Two flips φ, ψ for (X, σ_X) are conjugate if and only if there is an automorphism Φ of (X, σ_X) so that $\Phi\varphi = \psi\Phi$. It is clear, for a shift-flip system (X, σ_X, φ), that each $\sigma_X^n\varphi$ $(n \in \mathbb{Z})$ is a flip for (X, σ), and that $\sigma_X^{2n+1}\varphi$ and $\sigma_X^{2n}\varphi$ are conjugate to $\sigma_X\varphi$ and φ, respectively (Section 2.2).

The second goal of this thesis is inspired by [KimLP]. They showed that if a finite alphabet $\mathcal{A}$ has at least two elements, then the full $\mathcal{A}$-shift $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ has infinitely many non-conjugate flips. It is natural to ask whether this property hold for general shift spaces:

(P) *If (X, σ_X) has a flip, then it has infinitely many non-conjugate ones.*

Since some shift spaces have no flips (Section 2.3), we need the assumption that a shift space has a flip. We show that the property (P) hold for every infinite synchronized system (Theorem 6.1.1), but it is not true for some coded system (Section 6.2). In these cases, the shift spaces have many periodic points. On the other hand the Morse shift, which has no periodic points, does not satisfy the property (P) (Theorem 6.3.1).

Suppose that a shift-flip system (X, σ_X, φ) satisfies the property (P), and that $\varphi_1, \varphi_2, \cdots \varphi_{2n}$ are $2n$ flips for (X, σ_X). It is then obvious that the composition function $\varphi_1\varphi_2 \cdots \varphi_{2n}$ is an automorphism of (X, σ_X). Hence the automorphism group of a shift space with (P) is complicated: any infinite synchronized system retains the complexity, but the complexity is lost by passing to coded systems and minimal shifts [ChoK, Remarks].

This thesis is organized as follows.

In Chapter 2, we review some notions concerning shift spaces. We will use the same notations as in [LinM]. We collect definitions and properties of a shift-flip system from [ChoK, KimLP]. In addition we take a brief look at notions which are used in Chapter 4 and 5. These notions are contained in [BlaH, FieF1, Pet].

Chapter 3 is concerned with characterizations of sofic-like shifts (coded systems, synchronized systems and almost sofic shifts). Coded systems and synchronized systems are introduced in [BlaH]. A coded system is defined by a set of blocks with some conditions [BlaH]; and is also defined by a countable labeled graph [LinM]. We show that these definitions are equivalent, and there are other equivalences (Theorem 3.1.2). We show in Theorem 3.1.4 that both irreducible sofic shifts and synchronized systems form proper subclasses of coded systems. We conclude Section 3.1 with counterexamples of the reverse implications in Theorem 3.1.4. Every sofic shift is almost sofic in the sense of [LinM]: we use sofic subshifts to approximate the entropy. We prove in Section 3.2 that a sofic shift is almost sofic by using subshifts of finite type instead of sofic subshifts.

In Chapter 4, we discuss a disk system which is constructed in [Pet]. A disk system is presented by a labeled graph whose vertices and edges are contained in a disk with radius c. If c is sufficiently large, then the the disk system has positive entropy and is periodic saturated, but is not almost sofic. However the construction which is provided in [Pet] is not clear, and accordingly we make a detailed explanation. We modify the construction and obtain a disk system which still satisfies the properties as in [Pet]. Moreover, the disk system is irreducible, periodic points dense and has a flip, but is neither coded nor almost sofic (Theorem 4.0.1).

Chapter 5 contains a discussion of a coded system which is not almost sofic. We obtain formulas for the entropy and the zeta function of S-gap shifts (Section 5.1). In Section 5.2, we construct an irreducible shift space X having no periodic points and an S-gap shift Y with $h(Y) < h(X)$ by using results in Section 5.1, then combine X with Y to obtain a coded system Z. The coded system Z is not periodic saturated, so that it is not almost sofic (Remark 2.2.2). It also shows that the density of periodic points does not guarantee that a shift space is periodic saturated. In Section 5.3 we generalize the construction given in Section 5.2. Using the generalization given in Section 5.3 we construct an almost sofic shift which is not periodic points dense in Section 5.4.

In Chapter 6, we prove that every infinite synchronized system has the property (P). For this, we use a version of the method of 'markers' (Lemma 6.1.6). The fact that there is a finitary block play an important role in the construction of markers. Some coded systems do not have the property (P). The coded system constructed in Section 6.2 is introduced in [FieF2]. They constructed a coded system whose automorphism group is generated by the shift map, so the automorphism group is isomorphic to $\mathbb{Z}$ [FieF2, Corollary 2.2]. However it is not clear whether the coded system has a flip. Fortunately we can simplify the construction of [FieF2] and show that the coded system has a flip ρ defined by $\rho(x)_i = x_{-i}$. The Morse shift M is a minimal shift defined by the Morse sequence [MorH, LinM]. From [GotH, MorH] the Morse shift M can be presented by forbidden blocks, that is, M is the set of points in $\{0,1\}^{\mathbb{Z}}$ which has no blocks in the set $\{awawa : a \in \{0,1\}$ and $w \in \mathcal{B}(\{0,1\}^{\mathbb{Z}})\}$. The restraint is invariant under ρ, so that the Morse shift has a flip. The Morse shift does not have the property (P) from the result about the automorphism group of the Morse shift given in [Cov].

Chapter 2

Preliminaries

In this chapter, we review some definitions and properties related to shift spaces and flips for those spaces. Our references include [BlaH, ChoK, KimLP, LinM].

2.1 Shift spaces

Let $\mathcal{A}$ be a finite set. We call $\mathcal{A}$ an *alphabet*, and elements of $\mathcal{A}$ are called *symbols*. Let $\mathcal{A}^* = \bigcup_{n=0}^{\infty} \mathcal{A}^n$ and $\mathcal{A}^0 = \{\epsilon\}$ where ϵ is *the empty block*. Similarly for a subset $\mathcal{W}$ of $\mathcal{A}^*$, let $\mathcal{W}^* = \bigcup_{n=0}^{\infty} \mathcal{W}^n$.

The full $\mathcal{A}$-shift $\mathcal{A}^{\mathbb{Z}}$ is the set of all bi-infinite sequences of symbols from $\mathcal{A}$. A point $x \in \mathcal{A}^{\mathbb{Z}}$ is denoted by $x = \langle x_i \rangle_{i \in \mathbb{Z}} = \cdots x_{-2}x_{-1}.x_0x_1x_2\cdots$. The shift map σ on $\mathcal{A}^{\mathbb{Z}}$ shifts x one place to the left: $\sigma(x)_i = x_{i+1}$ for all $i \in \mathbb{Z}$. We give the discrete topology to $\mathcal{A}$, then $\mathcal{A}^{\mathbb{Z}}$ is compact in the product topology, and σ is a homeomorphism of $\mathcal{A}^{\mathbb{Z}}$. Let $x \in \mathcal{A}^{\mathbb{Z}}$ and $i, j \in \mathbb{Z}$. Then $x_{[i,j]} = x_i x_{i+1} \cdots x_j$ is called a block. The length of $x_{[i,j]}$ is $j - i + 1$ and denoted by $|x_{[i,j]}|$. If $i > j$, then $x_{[i,j]} = \epsilon$ with $|x_{[i,j]}| = 0$.

A subset X of $\mathcal{A}^{\mathbb{Z}}$ is a *shift space* (or simply *shift*) if it is closed and shif-invariant. The shift map on X is the restriction σ_X of σ on X. If X and Y are shift spaces and $X \subseteq Y$, then X is a subshift of Y. A shift space is also defined by a subset of $\mathcal{A}^*$: X is a shift space if there is a subset $\mathcal{F}$ of $\mathcal{A}^*$ so that every point in X has no blocks in $\mathcal{F}$. In this case we write $X = \mathbf{X}_{\mathcal{F}}$. If $\mathcal{F}$ is the empty set, then $\mathbf{X}_{\mathcal{F}} = \mathcal{A}^{\mathbb{Z}}$. If $\mathcal{F}$ is finite, then $\mathbf{X}_{\mathcal{F}}$ is said to be a *shift of finite type*. A shift space X has a finite type if and only if there is a non-negative integer M such that whenever uv, vw are allowed in X and $|v| \geq M$ then uvw is allowed in X [LinM, Theorem 2.1.8]. In this case X is said to be an M-step shift of finite type.

Let S be a subset of $\{0, 1, 2, \cdots\}$. We will denote by $\mathbf{X}(S)$ the closure of the set of all bi-infinite concatenations of blocks from $\{10^s : s \in S\}$. Then $\mathbf{X}(S)$ is a subshift of $\{0,1\}^{\mathbb{Z}}$ since $\mathbf{X}(S) = \mathbf{X}_{\mathcal{F}}$ where $\mathcal{F} = \{10^t1 : t \notin S\}$. We call $\mathbf{X}(S)$ the *S-gap shift*. The point 0^∞ belongs to $\mathbf{X}(S)$ when S is infinite. If $S = \{0, 1, 2 \cdots\}$ then $\mathbf{X}(S)$ is the full $\{0,1\}$-shift.

Suppose that X is a shift space. For each $n \in \mathbb{N}$, let $\mathcal{B}_n(X)$ be the set of n-blocks (blocks of length n) appearing in a point $x \in X$. The *language* $\mathcal{B}(X)$ of X is the union of $\mathcal{B}_n(X)$ for all n: $\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X)$. The following show that when a set of blocks is the language of a shift space; and that the language determines a shift space. For the proofs we refer the reader to [LinM, Section 1.3]

Proposition 2.1.1.

(1) *Let X be a shift space and $\mathcal{B}(X)$ be the language of X. If $w \in \mathcal{B}(X)$, then*

  (i) *every subblock of w belongs to $\mathcal{B}(X)$, and*

  (ii) *there are nonempty blocks $u, v \in \mathcal{B}(X)$ such that $uwv \in \mathcal{B}(X)$.*

(2) *Let $\mathcal{L} \subseteq \mathcal{A}^*$. Then $\mathcal{L}$ is the language of a shift space if and only if $\mathcal{L}$ satisfies the above condition* (1).

(3) *Let X and Y be subshifts of $\mathcal{A}^{\mathbb{Z}}$. If $\mathcal{B}(X) = \mathcal{B}(Y)$, then $X = Y$.*

(4) *Let X be a subshift of $\mathcal{A}^{\mathbb{Z}}$. Then X is a shift space if and only if whenever $x \in \mathcal{A}^{\mathbb{Z}}$ and each $x_{[i,j]} \in \mathcal{B}(X)$ then $x \in X$.*

A set $\mathcal{U}$ of blocks over $\mathcal{A}$ is *uniquely decipherable* if whenever $u_1u_2 \cdots u_k = v_1v_2 \cdots v_n$ with $u_i, v_j \in \mathcal{U}$, then $k = n$ and $u_i = v_i$ for $1 \leq i \leq n$. If $\mathcal{U}$ is uniquely decipherable, then the empty block ϵ does not belong to $\mathcal{U}$.

A shift space X is *irreducible* if for every ordered pair of $u, v \in \mathcal{B}(X)$ there is a block $w \in \mathcal{B}(X)$ such that $uwv \in \mathcal{B}(X)$.

Let (X, σ_X) be a shift space and m, n be integers with $-m \leq n$. Let $\Phi : \mathcal{B}_{m+n+1} \to \mathcal{A}$ assign to each block $w \in \mathcal{B}_{m+n+1}$ a symbol $\Phi(w)$ from $\mathcal{A}$ (we call Φ a block map). A map $\Phi_\infty : X \to \mathcal{A}^{\mathbb{Z}}$ is a *sliding block code* if $\Phi_\infty(x)_i = \Phi(x_{[i-m,i+n]})$ for all $x \in X$ and $i \in \mathbb{Z}$. If (Y, σ_Y) is a shift space and $\Phi_\infty : X \to Y$ is a sliding block code, then Φ_∞ is continuous and shift-commuting ($\Phi_\infty \circ \sigma_X = \sigma_Y \circ \Phi_\infty$). Conversely, if a map from a shift space to another shift is continuous and shift-commuting, then the map can be presented by a block map [Hed]. We often use the notation Φ instead of Φ_∞ when no confusion can arise. A sliding block code $\Phi : X \to Y$ is a *factor map* if Φ is onto. In this case we call Y a *factor* of X. A *conjugacy* from (X, σ_X) onto (Y, σ_Y) is a sliding block code which is one-to-one and onto. An *automorphism* of X is a conjugacy from X to itself.

Remarks 2.1.2. (1) For a sliding block code $\Phi : (X, \sigma_X) \to (\mathcal{A}^{\mathbb{Z}}, \sigma)$, the image $\Phi(X)$ is a subshift of $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ [LinM, Theorem 1.5.13].

(2) Let n be a positive integer. We define $\Psi_n : X \to (\mathcal{B}_n(X))^{\mathbb{Z}}$ by $\Psi_n(x)_i = x_{[i,i+n-1]}$. Then $\Psi_n(X)$ is a shift space [LinM, Proposition 1.4.3] and we call $X^{[n]} := \Psi_n(X)$ the nth higher block system of X. Furthermore, the nth higher block code Ψ_n is a conjugacy so that X and $X^{[n]}$ are conjugate [LinM, Example 1.5.10].

We define the *entropy* $h(X)$ of X from the numbers $|\mathcal{B}_n(X)|$, $n \geq 1$:

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{B}_n(X)|.$$

The entropy is an important number because it is invariant under conjugacy. For $n \in \mathbb{N}$, let $P_n(X)$ be the set of n-periodic points(i.e., $\sigma_X^n(x) = x$) of X. The cardinality of $P_n(X)$ is also conjugacy invariant. We use these values to define the following two notations. The *zeta function* $\zeta_X(t)$ of X is defined by

$$\zeta_X(t) = \exp\left(\sum_{n=1}^{\infty} \frac{|P_n(X)|}{n} t^n\right).$$

If $P(X) = \bigcup_{n \in \mathbb{N}} P_n(X)$ has sufficiently many elements so that

$$h(X) = \limsup_{n \to \infty} \frac{1}{n} \log |P_n(X)| =: h_p(X),$$

then X is said to be *periodic saturated* [Pet]. Observe that a point $x \in P_n(X)$ is uniquely determined by $x_{[0,n-1]}$, so that $|P_n(X)| \leq |\mathcal{B}_n(X)|$ and $h_p(X) \leq h(X)$. Every irreducible sofic shift is periodic saturated [LinM, Theorem 4.3.6].

2.2 Sofic-like shifts

Let $G = (\mathcal{V}, \mathcal{E})$ be a graph where $\mathcal{V}$ is the vertex set and $\mathcal{E}$ is the edge set of G. A graph G is a *countable graph* if both $\mathcal{V}$ and $\mathcal{E}$ are countable sets. If both $\mathcal{V}$ and $\mathcal{E}$ are finite, then G is a finite graph. If $\mathcal{L}$ is a labeling of the edges by a finite alphabet, then $\mathcal{G} = (G, \mathcal{L})$ is called a (*countable*) *labeled graph*. We use $\mathcal{L}$ to label paths and bi-infinite walks on G. For a path $\pi = e_1 e_2 \cdots e_k$ and a bi-infinite walk $x = \langle x_i \rangle_{i \in \mathbb{Z}}$ on G, we define

$$\mathcal{L}(\pi) = \mathcal{L}(e_1) \cdots \mathcal{L}(e_k) \quad \text{and} \quad \mathcal{L}_\infty(x) = \langle \mathcal{L}(x_i) \rangle_{i \in \mathbb{Z}}.$$

We call $\mathcal{L}(\pi)$ the label of π, and $\mathcal{L}_\infty(x)$ the label of x. Analogously we define $\mathcal{L}_\infty(\xi)$ where ξ is a right-infinite walk or left-infinite walk on G. A labeled graph $(G, \mathcal{L})$ is *right-resolving* if whenever two edges e and f start at the same vertex and $\mathcal{L}(e) = \mathcal{L}(f)$, then $e = f$.

If G is finite, we obtain two subshifts $\widehat{\mathbf{X}}_G$ and $\mathbf{X}_G$ of $\mathcal{V}^{\mathbb{Z}}$ and $\mathcal{E}^{\mathbb{Z}}$, respectively. We call $\widehat{\mathbf{X}}_G$ a vertex shift and $\mathbf{X}_G$ an edge shift. Both shifts are shifts of finite type.

Suppose that G is a countable graph and that $\mathcal{L}$ is a labeling of the edges by $\mathcal{A}$. In this thesis, we will be interested in $\mathbf{X}_G$ with $\mathcal{L}$. Then the set $\mathcal{L}_\infty(\mathbf{X}_G)$ is shift-invariant, but not closed in general. Let $\mathbf{X}_{(G,\mathcal{L})} = \overline{\mathcal{L}_\infty(\mathbf{X}_G)}$, then $\mathbf{X}_{(G,\mathcal{L})}$ is a subshift of $\mathcal{A}^{\mathbb{Z}}$. In this case, we call $(G, \mathcal{L})$ a *presentation* of $\mathbf{X}_{(G,\mathcal{L})}$. If G is finite, then $\mathbf{X}_{(G,\mathcal{L})} = \mathcal{L}_\infty(\mathbf{X}_G)$ and it is said to be a *sofic shift*. Every shift space X has a presentation $(G, \mathcal{L})$: we define a countable graph G and a labeling $\mathcal{L}$ of the edges by $\mathcal{B}_1(X)$. For each $w \in \mathcal{B}(X)$ we have a path labeled

w of length $|w|$. If π and τ are two paths labeled w and u respectively and $wu \in \mathcal{B}(X)$, then $\pi\tau$ is a path on G. Since $\mathcal{B}(X)$ is countable, so is G. It is clear that $X = \mathbf{X}_{(G,\mathcal{L})}$.

A graph $G = (\mathcal{V}, \mathcal{E})$ is irreducible if for every pair of vertices $I, J \in \mathcal{V}$ there is a path which starts at I and ends at J. A *coded system* is a shift space which is presented by an irreducible countable labeled graph. This system is introduced in [BlaH]. In that paper, a coded system X is defined by the language $\mathcal{B}(X)$ which is coded: $\mathcal{B}(X)$ is the set of subblocks of a block in $\mathcal{U}$ where $\mathcal{U}$ is uniquely decipherable and for any ordered pair u and v of $\mathcal{U}$, u is not a prefix of v.

A block $w \in \mathcal{B}(X)$ is *finitary* (or *intrinsically synchronizing*) for (X, σ_X) if whenever $uw, wv \in \mathcal{B}(X)$ then $uwv \in \mathcal{B}(X)$. It is clear that every subshift of finite type has a finitary block. In [Kit, Observation 6.1.5] it is shown that every sofic shift has a finitary block.

Proposition 2.2.1.

(1) *Every block containing a finitary block is also finitary.*

(2) *If (X, σ_X) has a finitary block and is conjugate to (Y, σ_Y), then (Y, σ_Y) is also has a finitary block.*

Proof. (1) Suppose that f is a finitary block for (X, σ_X) and $w = w'fw'' \in \mathcal{B}(X)$. Let $uw, wv \in \mathcal{B}(X)$. Then $uw = uw'fw''$ and $wv = w'fw''v$. Since $uw'f, fw''v \in \mathcal{B}(X)$ and f is finitary, we obtain $uw'fw''v \in \mathcal{B}(X)$, so that $uwv = uw'fw''v \in \mathcal{B}(X)$. Hence w is a finitary block for (X, σ).

(2) Suppose that $\Phi_\infty : X \to Y$ is a conjugacy with its inverse Ψ_∞. There is $N \geq 1$ such that $\Phi_\infty(x)_0 = \Phi(x_{[-N,N]})$ and $\Psi_\infty(y)_0 = \Psi(y_{[-N,N]})$ for all $x \in X, y \in Y$. Then

$$\Phi(\Psi(uwv)) = w \qquad (uwv \in \mathcal{B}(Y),\ |u| = |v| = 2N). \tag{2.1}$$

Let f be a finitary block for (X, σ_X). We may assume that $|f| = 2n+1$ for some $n > N$ by (1). Let $x \in X$ with $x_{[-n,n]} = f$, $n + N = M$ and $\Phi_\infty(x)_{[-M,M]} = w$.

It is clear that $\Psi(w) = f$. We claim that w is a finitary block for (Y, σ_Y). Let uw, $wv \in \mathcal{B}(Y)$. There are $y, y' \in Y$ and $i, j \in \mathbb{Z}$ such that

$$y_{[-i,M]} = uw, \quad y'_{[-M,j]} = wv, \quad y_{[-M,M]} = y'_{[-M,M]} = w.$$

Then

$$\begin{aligned}\mathcal{B}(Y) \ni \Psi_\infty(y)_{[-i-N,M-N]} &= \Psi(y_{[-i-2N,M]}) = \Psi(y_{[-i-2N,-M-1+N]})\Psi(y_{[-M,M]}) \\ &= \Psi(y_{[-i-2N,-M-1+N]})\Psi(w) = \Psi(y_{[-i-2N,-M-1+N]})f\end{aligned}$$

and similarly, $\mathcal{B}(Y) \ni \Psi_\infty(y')_{[-M+N,j+N]} = f\Psi(y'_{[M+1-N,j+2N]})$. Since f is finitary, there is $z \in X$ with $z_{[-i-N,j+N]} = \Psi_\infty(y)_{[-i-N,M-N]}\Psi_\infty(y')_{[M-N+1,j+N]}$. It is straightforward to compute the value

$$\begin{aligned}\Phi(z_{[-i-N,j+N]}) &= \Phi\big(\Psi(y_{[-i-2N,M]})\big)\Phi\big(\Psi(y'_{[M-4N+1,j+2N]})\big) \\ &= y_{[-i,M-2N]}y'_{[M-2N+1,j]} = uwv\end{aligned}$$

because $n > N$, $y_{[-M,M]} = y'_{[-M,M]}$ and (2.1). Since $\Phi(z_{[-i-N,j+N]}) = \Phi_\infty(z)_{[-i,j]}$ belongs to $\mathcal{B}(Y)$, we obtain the desired result. $\square$

A shift space X is a *synchronized system* if it is irreducible and has a finitary block. Synchronized systems form a subclass of coded systems [BlaH]. A shift space X is *almost sofic* if there are sofic subshifts X_n of X such that $\lim_{n\to\infty} h(X_n) = h(X)$. Thus it is clear that every shift of finite type and every sofic shift are almost sofic. In [Pet] almost sofic shift X is defined in terms of shifts X_n of finite type.

Remark 2.2.2. An almost sofic shift is periodic saturated. Let X be an almost sofic shift and $\epsilon > 0$. We show that

$$-\epsilon + h(X) < h_p(X) \le h(X).$$

The right-hand inequality hold by definition. Since X is almost sofic, there are sofic subshifts X_k of X such that $\lim_{k\to\infty} h(X_k) = h(X)$. Then there is a $K \ge 1$ so that $-\epsilon + h(X) < h(X_K)$. Since $P_n(X_k) \subseteq P_n(X)$ for all k, n, and since each X_k is periodic saturated (Section 2.1), we have

$$-\epsilon + h(X) < h(X_K) = h_p(X_K) \leq h_p(X)$$

then the proof is done [Pet].

For a synchronized system X, the *synchronized entropy* $h_s(X)$ is defined as follows [Tho]: let f be a finitary block for (X, σ_X). For each integer $n \geq 1$, let $\mathcal{W}(f, n)$ be the set of all blocks $w \in \mathcal{B}_n(X)$ such that $fwf \in \mathcal{B}(X)$. Then

$$h_s(X) := \limsup_{n\to\infty} \frac{1}{n} \log |\mathcal{W}(f, n)|.$$

This value is independent of the choice of finitary block: let g be another finitary block for (X, σ_X). Since X is irreducible and f, g are finitary, there are two positive integers M_1, M_2 such that

$$|\mathcal{W}(f, n)| \leq |\mathcal{W}(g, n + M_1)| \quad \text{and} \quad |\mathcal{W}(g, n)| \leq |\mathcal{W}(f, n + M_2)| \quad (n \geq 1).$$

Therefore

$$\limsup_{n\to\infty} \frac{1}{n} \log |\mathcal{W}(f, n)| = \limsup_{n\to\infty} \frac{1}{n} \log |\mathcal{W}(g, n)|,$$

and $h_s(X)$ is independent of the choice of finitary block. It is clear that $h_s(X) \leq h(X)$ since $\mathcal{W}(f, n) \subseteq \mathcal{B}_n(X)$. For every irreducible sofic shift X, $h_s(X) = h(X)$ [Tho, Lemma 3.1]. If $h_s(X) = h(X)$, then X is periodic saturated since each $w \in \mathcal{W}(f, n)$ induces an $(n + |f|)$-periodic point in X.

Remark 2.2.3. Generally we have

$$h_s(X) \leq h_p(X) \leq h(X). \tag{2.2}$$

The values $h_s(X)$, $h_p(X)$ and $h(X)$ are the same for an irreducible sofic shift X. We conjecture that there is a shift space such that three values $h_s(X)$, $h_p(X)$ and $h(X)$ are all distinct.

There are examples such that $h_s(X) < h(X)$. In [Pet, Example 3.2], there is a synchronized system X such that $h(X) > \log \lambda = h_s(X)$ where λ is the golden mean number. In Section 5.2, we construct a synchronized system Y which is not periodic saturated. By equation (2.2), we obtain $h_s(Y) < h(Y)$.

2.3 Flips for a shift space

Let (X, σ_X) be a shift space over $\mathcal{A}$. A homeomorphism $\varphi : X \to X$ is said to be a *flip* for (X, σ) if

$$\varphi\sigma_X = \sigma_X^{-1}\varphi \qquad \text{and} \qquad \varphi^2 = \mathrm{id}_X.$$

In this case we say that (X, σ_X, φ) is a *shift-flip system*. We define the *mirror map* $\rho : X \to \mathcal{A}^{\mathbb{Z}}$ by

$$\rho(x)_i = x_{-i} \qquad (x \in X,\ i \in \mathbb{Z}).$$

If X is closed under ρ, then ρ is a flip for (X, σ_X). Let (X, σ_X, φ) and (Y, σ_Y, ψ) be two shift-flip systems. If there is a homeomorphism $\Phi : X \to Y$ such that

$$\Phi \circ \sigma_X = \sigma_Y \circ \Phi \qquad \text{and} \qquad \Phi \circ \varphi = \psi \circ \Phi,$$

then we say that (X, σ_X, φ) and (Y, σ_Y, ψ) are *conjugate*, and Φ is called a *conjugacy* from (X, σ_X, φ) to (Y, σ_Y, ψ). If (X, σ_X, φ) and (X, σ_X, ψ) are conjugate, we say that φ and ψ are conjugate. Thus φ and ψ are conjugate if and only if there is an automorphism Φ of (X, σ_X) such that $\Phi\varphi = \psi\Phi$.

Not every shift space has a flip since a flip for (X, σ) is a conjugacy from (X, σ_X) to (X, σ_X^{-1}). There is a shift space (X, σ_X) which do not conjugate to (X, σ_X^{-1}) [LinM, Examples 7.4.19 and 12.3.2]. However if there is a flip φ for (X, σ_X), then it is easily seen that the maps $\varphi\sigma_X^m$, $m \in \mathbb{Z}$, are flips for (X, σ_X), and that they are all distinct whenever X is infinite. If $m - n$ is even, then $\varphi\sigma_X^m$ and $\varphi\sigma_X^n$ are conjugate since $\sigma_X^{(m-n)/2}$ is a conjugacy from $(X, \sigma_X, \varphi\sigma_X^m)$ to $(X, \sigma_X, \varphi\sigma_X^n)$. Thus $\varphi\sigma_X^m$ is conjugate to φ or $\varphi\sigma_X$.

For a shift-flip system (X, σ_X, φ), let

$$F(\varphi, n) = \{x : \sigma_X^n(x) = \varphi(x) = x\} \qquad (n = 1, 2, 3, \cdots).$$

It is easy to show that $|F(\varphi, n)| < \infty$ for all n, and that $|F(\varphi, n)| = |F(\psi, n)|$ whenever φ and ψ are conjugate.

Chapter 3

Coded systems and almost sofic shifts

In this chapter we consider coded systems, synchronized systems and almost sofic shifts which generalize sofic shifts.

3.1 Characterizations of coded systems

In this section we study equivalent characterizations of coded systems, some properties of irreducible sofic shifts and coded systems (Theorem 3.1.2 and 3.1.4). These are stated in [LinM, Section 13.5] without proofs; but they presented references to prove them. Here we provide detailed proofs.

Definition 3.1.1. Let $\mathcal{U}$ be a countable subset of $\mathcal{A}^*$. We define $\mathbf{X}(\mathcal{U})$ to be the closure of the set of all bi-infinite concatenations of blocks from $\mathcal{U}$. It is clear that $\mathbf{X}(\mathcal{U})$ is a subshift of $\mathcal{A}^{\mathbb{Z}}$.

Theorem 3.1.2. *Let X be a shift space over $\mathcal{A}$. The following are equivalent.*

(1) *X is a coded system.*

(2) *X has an irreducible right-resolving presentation.*

(3) *There is a subset* $\mathcal{U}$ *of* $\mathcal{A}^*$ *such that* $\mathcal{U}$ *is uniquely decipherable and* $X = \mathbf{X}(\mathcal{U})$.

(4) X *contains an increasing sequence of irreducible shift of finite type whose union is dense in* X.

(5) *There is a subset* $\mathcal{C}$ *of* $\mathcal{A}^*$ *such that* $X = \mathbf{X}(\mathcal{C})$.

Remark 3.1.3. In the case (5) we call $\mathcal{C}$ a *code* for X. It is clear that for every set $\mathcal{C}$ of blocks there is a unique coded system for which $\mathcal{C}$ is a code.

Proof. Let X be a shift space over $\mathcal{A}$.

(1) $\Rightarrow$ (2)([FieF1]) Let X be a coded system. There is an irreducible presentation $(G, \mathcal{L})$ of X. Let I be a vertex of G. Consider the set E of labels of right-infinite walks starting at I on G. By the irreducibility of G, there is a shortest path labeled w from I to itself. Clearly, the right-infinite sequence w^∞ belongs to E. There are two cases. If $E = \{w^\infty\}$, then the labeled graph consisting of only w is an irreducible right-resolving presentation of X. Otherwise we proceed as follows. There are two paths $e_1 \cdots e_n$ and $f_1 \cdots f_n$ starting at I such that $\mathcal{L}(e_1 \cdots e_{n-1}) = \mathcal{L}(f_1 \cdots f_{n-1})$ but $\mathcal{L}(e_n) \neq \mathcal{L}(f_n)$. Let F be the set of all finite paths from I to itself. Since F is countable, we write $F = \{\gamma_i : i \in \mathbb{Z}\}$. There are two paths π, τ such that $e_1 \cdots e_n \pi, f_1 \cdots f_n \tau \in F$ because of the irreducibility of G. For $i \in \mathbb{Z}$, let

$$w_i = \mathcal{L}(\gamma_i e_1 \cdots e_{n-1}), \quad u = \mathcal{L}(e_n \pi), \quad u' = \mathcal{L}(f_n \tau).$$

Note that $w_i = \mathcal{L}(\gamma_i f_1 \cdots f_{n-1})$ since $\mathcal{L}(e_1 \cdots e_{n-1}) = \mathcal{L}(f_1 \cdots f_{n-1})$. Then any finite concatenation

$$w = w_{i_1} v_{i_1} \cdots w_{i_k} v_{i_k} \in \mathcal{B}(X) \tag{3.1}$$

for all $v_{i_j} \in \{u, u'\}$ and $i_j \in \mathbb{Z}$ since w is a label of a finite concatenation of blocks in F. If we put $x = \langle w_i u \rangle_{i \in \mathbb{Z}}$ such that $x_{[0,\infty)}$ begins with w_0, then $x \in X$ and $\{\sigma_X^n(x) : n \in \mathbb{Z}\}$ is dense in X from (3.1) and the irreducibility of G.

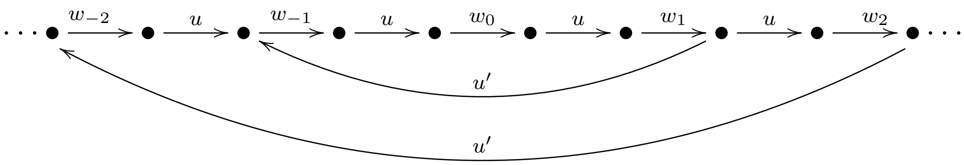

Figure 3.1: The labeled graph $\mathcal{H}$

We now construct a labeled graph $\mathcal{H} = (H, \mathcal{L}')$ as follows (Figure 3.1). Let

$$\mathbb{Z} \cup \{n_j : n \in \mathbb{N}, \quad 1 \leq j \leq l-1\}$$

be the vertex set of H where $l = |u'|$. For each $i \in \mathbb{Z}$, we draw an edge h_i labeled x_i from i to $i+1$. For each $n \in \mathbb{N}$, we draw a path $g_1 g_2 \cdots g_l$ labeled u' from $|w_0 u \cdots w_{n-1} u w_n|$ to $-|w_{-n} u \cdots w_{-1} u|$ such that each g_j ends at n_j, $1 \leq j \leq l-1$. This yields the labeled graph $\mathcal{H} = (H, \mathcal{L}')$, i.e., $\mathcal{H}$ consists of an bi-infinite walks $\xi = \langle \xi_i \rangle_{i \in \mathbb{Z}}$ labeled x and countably many paths μ_n labeled u' from ξ_m to ξ_{-k} for some $m, k \in \mathbb{N}$ (the values m, n depend on n), and also $\mathcal{L}' = \mathcal{L}$. It is obvious that H is irreducible and $\mathbf{X}_{\mathcal{H}} \subseteq X$. If J is a vertex of H and there are two edges e, f starting at J, then $\mathcal{L}(e)$ and $\mathcal{L}(f)$ are the first symbol of u and u', respectively by the construction. Thus $\mathcal{L}(e) = \mathcal{L}(e_n) \neq \mathcal{L}(f_n) = \mathcal{L}(f)$, and $\mathcal{H}$ is right-resolving. Since $x \in \mathbf{X}_{\mathcal{H}}$ and $\{\sigma_X^n(x) : n \in \mathbb{Z}\}$ is dense in X, $X \subseteq \mathbf{X}_{\mathcal{H}}$. Therefore $\mathcal{H}$ is an irreducible right-resolving presentation of X.

(2) $\Rightarrow$ (3)([BlaH]) Suppose that $\mathcal{G} = (G, \mathcal{L})$ is an irreducible right-resolving presentation of X. Let I be a vertex of G. We define U to be the set of paths π from I to itself but π does not pass through I. Let $\mathcal{U} = \mathcal{L}(U) = \{\mathcal{L}(\pi) : \pi \in U\}$. We will show that $\mathcal{U}$ is uniquely decipherable and $X = \mathbf{X}(\mathcal{U})$.

To show that $\mathcal{U}$ is uniquely decipherable let $u_1 u_2 \cdots u_m = v_1 v_2 \cdots v_n$ where $u_i, v_j \in \mathcal{U}$. There are two paths $\pi_1 \pi_2 \cdots \pi_m$ and $\tau_1 \tau_2 \cdots \tau_n$ such that $\pi_i, \tau_j \in U$, $\mathcal{L}(\pi_i) = u_i$ and $\mathcal{L}(\tau_j) = v_j$. If $|\pi_1| < |\tau_1|$, then we can write $\pi_1 = e_1 \cdots e_s$ and $\tau_1 = f_1 \cdots f_s f_{s+1} \cdots f_t$. Since $\mathcal{G}$ is right-resolving, $e_i = f_i$ for $1 \leq i \leq s$. Hence

f_s ends at I, and τ_1 cannot be in U. Similarly, if $|\pi_1| > |\tau_1|$ then $\pi_1 \notin U$. Thus $|\pi_1| = |\tau_1|$, and $\pi_1 = \tau_1$ since $\mathcal{G}$ is right-resolving. We get $u_1 = v_1$. Continuing this process we get $u_i = v_i$ and $m = n$.

To show that $\mathbf{X}_{\mathcal{G}} = \mathbf{X}(\mathcal{U})$, it is enough to show that $\mathbf{X}_{\mathcal{G}} \subseteq \mathbf{X}(\mathcal{U})$. Let $x \in \mathbf{X}_{\mathcal{G}}$ and let N be a positive integer. There is a path π on G such that $\mathcal{L}(\pi) = x_{[-N,N]}$. Since G is irreducible, there are two paths τ and τ' such that $\tau\pi\tau'$ starts and ends at I. Then $\tau\pi\tau'$ is a finite concatenation of blocks in U, and $\mathcal{L}(\tau\pi\tau') = \mathcal{L}(\tau)x_{[-N,N]}\mathcal{L}(\tau')$ is a finite concatenation of blocks in $\mathcal{U}$. Therefore there is $z \in \mathbf{X}(\mathcal{U})$ with $z_{[-N,N]} = x_{[-N,N]}$. Since N is arbitrary, $x \in \mathbf{X}(\mathcal{U})$.

(3) $\Rightarrow$ (4). Suppose that $\mathcal{U}$ is a uniquely decipherable set with $X = \mathbf{X}(\mathcal{U})$. We can write $\mathcal{U} = \{u_1, u_2, u_3, \cdots\}$. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{u_1, u_2, \cdots, u_n\}$ and $X_n = \mathbf{X}(\mathcal{U}_n)$. Then it is obvious that every X_n is irreducible, $X_n \subseteq X_{n+1}$ and $\bigcup_{n=1}^{\infty} X_n$ is dense in X. It remains to show that each X_n is a shift of finite type. For each n, there is a finite labeled graph $\mathcal{G}_n = (G_n, \mathcal{L}_n)$: fix a vertex I. We draw n paths labeled $u_1, u_2, \cdots, u_n$ from I to itself. It is clear that $\mathcal{G}_n$ is a presentation of X_n. Then $(\mathcal{L}_n)_\infty : \mathbf{X}_G \to \mathbf{X}_{\mathcal{G}_n} = X_n$ is a conjugacy, hence X_n is a shift of finite type [LinM, Theorem 2.1.10 and Proposition 2.2.6].

(4) $\Rightarrow$ (5)([Kri2]) Suppose that $X_1, X_2, \cdots$ are irreducible shifts of finite type, $X_1 \subseteq X_2 \subseteq \cdots$, and that $\overline{\bigcup_{n=1}^{\infty} X_n} = X$. Since each X_n has finite type, there are $L_1, L_2, \cdots \in \mathbb{N}$ and $\mathcal{F}_1, \mathcal{F}_2, \cdots \subseteq \mathcal{A}^*$ such that $L_1 \leq L_2 \leq \cdots$ and $\mathcal{F}_n$ is subset of $\mathcal{A}^{L_n}$ such that $X_n = \mathbf{X}_{\mathcal{F}_n}$.

Let $w \in \mathcal{B}(X_1)$ with $w^\infty \in X_1$, then $w^\infty \in X_n$ for all n. Let $n = 1, 2, \cdots$. We choose numbers $M_n \geq 1$ with $M_n|w| > L_n$ and define a subset $\mathcal{U}_n$ of $\mathcal{B}(X_n)$ such that

(i) every block in $\mathcal{B}(X_n)$ is a subblock of a block in $\mathcal{U}_n$, and

(ii) every block in $\mathcal{U}_n$ starts and ends in w^{M_n}

Since X_n is irreducible, $\mathcal{U}_n$ is non-empty. Let

$$\mathcal{C}_n = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \cdots \cup \mathcal{U}_n, \quad \text{and} \quad \mathcal{C} = \bigcup_{k=1}^{\infty} \mathcal{U}_k.$$

We will show that $\mathbf{X}(\mathcal{C}) = X$. We divide the proof into three steps.

[Step 1] *Let $k, m \in \mathbb{N}$ and $u \in (\mathcal{C}_n)^*$. Then $w^k u w^m \in \mathcal{B}(X_n)$*: We use induction on n. Let $u = v_1 v_2 \cdots v_k \in (\mathcal{C}_1)^*$. Suppose that a subblock u' of u belongs to $\mathcal{F}_1$. Since $|u'| = L_1$ and $M_1|w| > L_1$, (ii) implies that u' is a subblock of v_j for some j or w^M for some $M \geq M_1$. Then $v_j \notin \mathcal{B}(X_1)$ or $w^M \notin \mathcal{B}(X_1)$. It is a contradiction, so that $u \in \mathcal{B}(X_1)$.

Suppose that the statement is true for an n. Let $u = v_1 v_2 \cdots v_k \in (\mathcal{C}_{n+1})^*$. Suppose that a subblock u' of u belongs to $\mathcal{F}_{n+1}$. Then u' occurs in a block of $(\mathcal{C}_n)^*$ or $v_j \in \mathcal{U}_{n+1}$ for some j or w^M, $M \geq M_{n+1}$. In the first case, $u' \in \mathcal{B}(X_n) \subseteq \mathcal{B}(X_{n+1})$ since $(\mathcal{C}_n)^* \subseteq \mathcal{B}(X_n)$. In the second or third case, $v_j \notin \mathcal{B}(X_{n+1})$ or $w^M \notin \mathcal{B}(X_{n+1})$. In either case, it contradicts to the assumption, so that $u \in \mathcal{B}(X_{n+1})$.

[Step 2] *For each n $X_n = \mathbf{X}(\mathcal{C}_n)$*: By (i) and [Step 1], we get $\mathcal{B}(X_n) = \mathcal{B}(\mathcal{C}_n)$. By Proposition 2.1.1(3), $X_n = \mathbf{X}(\mathcal{C}_n)$.

[Step 3] *We have $\overline{\bigcup_{n=1}^{\infty} X_n} = \mathbf{X}(\mathcal{C})$, so that $X = \mathbf{X}(\mathcal{C})$*: Since $\mathcal{C}_n \subseteq \mathcal{C}$, [Step 2] implies that we obtain $\overline{\bigcup_{n=1}^{\infty} X_n} \subseteq \mathbf{X}(\mathcal{C})$. Let $x \in \mathbf{X}(\mathcal{C})$ and $M \geq 0$. There are blocks $u_1, u_2, \cdots, u_k \in \mathcal{C}$ such that $x_{[-M,M]}$ is a subblock of $u_1 u_2 \cdots u_k = v$. There is a positive integer N such that $u_1, u_2, \cdots, u_k \in \mathcal{C}_N$. Since $v \in (\mathcal{C}_N)^*$, [Step 1] implies that $x_{[-M,M]}$ is a subblock of a block in $\mathcal{B}(X_N)$, then $x_{[-M,M]} \in \mathcal{B}(X_N)$. Since M is arbitrary, $x \in X_N \subseteq \overline{\bigcup_{n=1}^{\infty} X_n}$.

(5) $\Rightarrow$ (1). Suppose that there is a subset $\mathcal{C}$ of $\mathcal{A}^*$ such that $X = \mathbf{X}(\mathcal{C})$. We can write $\mathcal{C} = \{w_1, w_2, \cdots\}$ since $\mathcal{C}$ is countable. We construct a labeled graph $\mathcal{G}$ as follows. Let I be a vertex. For each $n \in \mathbb{N}$, we draw a path labeled w_n from I to itself. Then $\mathcal{G}$ is a presentation of X by definition. □

Suppose that X is a shift space. We define

$$\pi_{(-\infty,-1]}(X) = \{x_{(-\infty,-1]} : x \in X\}, \quad \pi_{[0,\infty)}(X) = \{x_{[0,\infty)} : x \in X\}$$
$$F(\lambda, X) = \{\mu \in \pi_{[0,\infty)}(X) : \lambda.\mu \in X\} \quad (\lambda \in \pi_{(-\infty,-1]}(X)).$$

We call $F(\lambda, X)$ the *future* of λ. We define a labeled graph called the *future cover* of X as follows. The vertices are the futures of left-infinite sequences. Let $F(\lambda, X)$, $F(\lambda', X)$ be vertices and $a \in \mathcal{B}_1(X)$. There is an edge labeled a from $F(\lambda, X)$ to $F(\lambda', X)$ whenever $\lambda' = \lambda a$. We do this process for all vertices and elements in $\mathcal{B}_1(X)$, then we obtain the future cover of X [LinM, Kri1].

Analogously, for a block $w \in \mathcal{B}(X)$ we define $F(w, X)$ to be the set of $\mu \in \pi_{[0,\infty)}(X)$ such that there is a point $x \in X$ with $w\mu = x_{[-|w|,\infty)}$.

Theorem 3.1.4. *Let X be an irreducible shift space. We consider the four properties:*

(1) *X is a sofic shift.*

(2) *X has a countably many futures of left-infinite sequences.*

(3) *X has a finitary block.*

(4) *X is a coded system.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Proof. Let X be an irreducible shift space.

(1) $\Rightarrow$ (2)([Kri1]) Suppose that X is a sofic shift and $(G, \mathcal{L})$ is a finite presentation of X, i.e., $X = \mathcal{L}_\infty(\mathbf{X}_G)$. Let $\lambda \in \pi_{(-\infty,-1]}(X)$ and $e \in \mathcal{E}(G)$. We define $\mathcal{E}(\lambda)$ to be the set of $f \in \mathcal{E}(G)$ such that there is a left-infinite walk $\xi \in \pi_{(-\infty,-1]}(\mathbf{X}_G)$ which ends with f and is labeled by λ. Let $F(e, \mathbf{X}(G))$ denote the set of right-infinite walks $\eta \in \pi_{[0,\infty)}(\mathbf{X}_G)$ such that $e\eta \in \pi_{[0,\infty)}(\mathbf{X}_G)$.

By definition, $F(\lambda, X) \subseteq \bigcup_{e \in \mathcal{E}(\lambda)} \mathcal{L}_\infty(F(e, \mathbf{X}(G)))$. Conversely, suppose that $e \in \mathcal{E}(\lambda)$ and $\mu \in \mathcal{L}_\infty(F(e, \mathbf{X}(G)))$. Then there is $\eta \in F(e, \mathbf{X}(G))$ such that $\mathcal{L}_\infty(\eta) = \mu$ and $e\eta = ee_0e_1e_2\cdots$ is a right-infinite walk on G. Since $e \in \mathcal{E}(\lambda)$, there is $\xi = \cdots g_{-3}g_{-2}e$ on G such that $\mathcal{L}_\infty(\xi) = \lambda$. Then

$\xi.\eta = \cdots g_{-3}g_{-2}e.e_0e_1e_2\cdots$ is a bi-infinite walk on G and $\mathcal{L}_\infty(\xi.\eta) = \lambda.\mu$. Thus $\mu \in F(\lambda, X)$. We obtain $F(\lambda, X) = \bigcup_{e\in\mathcal{E}(\lambda)} \mathcal{L}_\infty(F(e, \mathbf{X}(G)))$. Thus $|\{F(\lambda, X) : \lambda \in \pi_{(-\infty,-1]}(X)\}|$ is less than or equal to the number of non-empty subsets of $\mathcal{E}(G)$.

(2) $\Rightarrow$ (3)([FieF1]) Suppose that $(G, \mathcal{L})$ is the future cover of X and $\mathcal{V}(G)$ is countable. Let Y be the set of all bi-infinite walks on G. If $\mathcal{V}(G)$ is finite, then $Y = \mathbf{X}_G$ is a shift space. Otherwise, Y is not a shift space in general. However $\mathcal{L}_\infty(Y) = X$ in either case.

For each vertex I, let Y_I be the set of bi-infinite walks $\langle y_i\rangle_{i\in\mathbb{Z}}$ on G such that y_0 starts at I. Then $Y = \bigcup_{I\in\mathcal{V}(G)} Y_I$ and $X = \bigcup_{I\in\mathcal{V}(G)} \mathcal{L}_\infty(Y_I)$. Since $\mathcal{V}(G)$ is countable, the Baire category theorem implies that there are a vertex I, a positive number N and a block $w \in \mathcal{B}_{2N+1}(X)$ such that $[w]_{-N} = \{x \in X : x_{[-N,N]} = w\} \subseteq \overline{\mathcal{L}_\infty(Y_I)}$.

The block w is finitary for (X, σ_X). Indeed, suppose that uw, $wv \in \mathcal{B}(X)$. Let x, x' be points of X with $x_{[-|u|-N,N]} = uw$ and $x'_{[-N,N+|v|]} = wv$. Let $M = N + \max\{|u|, |v|\}$. Since $x, x' \in [w]_{-N}$, There are two points $y, y' \in Y_I$ such that $\mathcal{L}_\infty(y) = x$ and $\mathcal{L}_\infty(y') = x'$. Then $y_{(-\infty,-1]}.y'_{[0,\infty)} = z$ is a bi-infinite walk on G since y_{-1} ends at I and y'_0 starts ate I. Thus $uwv = \mathcal{L}_\infty(z)_{[-|u|-N,N+|v|]} \in \mathcal{B}(X)$.

(3) $\Rightarrow$ (4). Suppose that f is a finitary block for (X, σ_X). By the irreducibility of X, there is the set $\mathcal{W}$ of blocks $fw \in \mathcal{B}(X)$ such that $fwf \in \mathcal{B}(X)$. It is clear that $\mathcal{W}^* \subseteq \mathcal{B}(X)$ since f is finitary. Hence $\mathcal{B}(\mathbf{X}(X)) \subseteq \mathcal{B}(X)$. By the irreducibility of X, $\mathcal{B}(X) \subseteq \mathcal{B}(\mathbf{X}(X))$. Thus $X = \mathbf{X}(\mathcal{W})$ by Proposition 2.1.1(3). Theorem 3.1.1 implies that X is a coded system. □

We now consider the reverse implications of Theorem 3.1.4. There are examples to show that these reverse implications can not hold. Example 3.1.5 shows that (2) does not imply that (1).

Example 3.1.5. ([FieF1]) Let $X = \mathbf{X}(\{0^n1^n : n = 1, 2, \cdots\})$. It is obvious that X is irreducible. For a block $w \in \mathcal{B}(X)$, the follower set of w is defined as

$$\mathcal{F}(w, X) = \{v \in \mathcal{B}(X) : wv \in \mathcal{B}(X)\}.$$

We consider $\mathcal{F}(10^k1, X)$, $k \geq 1$. Let $n \neq m$. We may assume that $n < m$. Since $1^{m-1} \in \mathcal{F}(10^m1, X) \backslash \mathcal{F}(10^{n-1}1, X)$, we obtain $\mathcal{F}(10^n1, X) \neq \mathcal{F}(10^m1, X)$; hence X has a infinite number of follower sets, and then X is not a sofic shift [LinM, Theorem 3.2.10].

Let $\lambda \in \pi_{(-\infty,-1]}(X)$. Then $\lambda = 1^\infty$ or $\lambda = 0^\infty$ or λ ends with 10^k for some $k \geq 1$ or λ ends with 10^k1^m for some $1 \leq m \leq k$. Suppose that λ and $\lambda' \in \pi_{(-\infty,-1]}(X)$ end with 10^k and $10^{k'}$, respectively. If $k \neq k'$, then $F(\lambda, X) \neq F(\lambda', X)$: we may assume that $k > k'$. A right-infinite sequence $\mu \in \pi_{[0,\infty)}(X)$ starting with 1^k0 is in $F(\lambda, X) \setminus F(\lambda', X)$. Similarly, we obtain $F(\lambda, X) \neq F(\lambda', X)$ whenever λ and $\lambda' \in \pi_{(-\infty,-1]}(X)$ end with 10^k1^m and $10^{k'}1^{m'}$ respectively, and $k \neq k'$ or $m \neq m'$. Thus the collection F of follower sets of left-infinite sequences is

$$\begin{aligned} F = &\{F(\lambda, X) : \lambda \text{ end with } 10^k \text{ for } k \geq 1\} \\ &\cup \{F(\lambda, X) : \lambda \text{ end with } 10^k1^m \text{ for } 1 \leq m \leq k\} \\ &\cup \{F(1^\infty, X), F(0^\infty, X)\}. \end{aligned}$$

As the above argument, $F \setminus \{F(1^\infty, X), F(0^\infty, X)\}$ is countable, so F is countable. Thus X has a countably many futures of left-infinite sequences, but is not a sofic shift.

The following example shows that (4)$\not\Rightarrow$(3) and (3)$\not\Rightarrow$(2). For this we use the property of the coded system in Section 6.2.

Example 3.1.6. In Section 6.2 we will construct a code $\mathcal{C}$ over $\{0, 1, 2\}$ such that the infinite coded system $X = \mathbf{X}(\mathcal{C})$ has only two non-conjugate flips. Then X is not a synchronized system (Theorem 6.1.1). Since X is irreducible, it has no finitary blocks. This proves that (4) $\not\Rightarrow$ (3) in Theorem 3.1.4.

Let $\mathcal{A} = \{0, 1, 2, 3\}$ and $\mathcal{C}' = \mathcal{C} \cup \{3\}$. It is clear that $\mathbf{X}(\mathcal{C}')$ is irreducible and 3 is a finitary symbol for $\mathbf{X}(\mathcal{C}')$, so that $\mathbf{X}(\mathcal{C}')$ is a synchronized system. Since X is a subset of $\mathbf{X}(\mathcal{C}')$ and $\{F(\lambda, X) : \lambda \in \pi_{(-\infty,-1]}(X)\}$ is uncountable, we obtain $\mathbf{X}(\mathcal{C}')$ does not satisfy (2) in Theorem 3.1.4 (The idea is similar to [FieF1, Corollary 1.3], [BlaH, Proposition 4.1] and [Mey, Corollary 2.17]).

3.2 Almost sofic shifts

Almost sofic shifts generalize sofic shifts in terms of entropy: an almost sofic shift X is a shift space such that for every $\delta > 0$ there is a sofic subshift Y of X with $|h(X) - h(Y)| < \delta$. In [Pet] we have a shift Y of finite type instead of a sofic shift. We will show in Theorem 3.2.1 that a sofic shift is almost sofic in the sense [Pet]. An almost sofic shift is periodic saturated (Remark 2.2.2), but is not closed under a factor map [Pet, Example 3.3]. Some almost sofic shifts are coded: for example, an irreducible sofic shift is both coded and almost sofic. In the last of this section we collect of examples which are both coded and almost sofic.

The first part of this section focuses on the proof of the following theorem:

Theorem 3.2.1. *Let X be a sofic shift over $\mathcal{A}$. Then there are subshifts $X_1, X_2, X_3, \cdots,$ of X such that each X_n has finite type and* $\lim_{n\to\infty} h(X_n) = h(X)$.

For the proof of Theorem 3.2.1 we will find certain subsets of $\mathcal{B}(X)$, and use the sets to construct subshifts of finite type and to compute the entropy $h(X)$. To compute the entropy $h(X)$ we define the entropy of a subset of $\mathcal{A}^*$. Suppose that $\mathcal{U}$ is a non-empty subset of $\mathcal{A}^*$ such that $\mathcal{U} \neq \{\epsilon\}$. The generating function $G_\mathcal{U}$ of $\mathcal{U}$ is defined by

$$G_\mathcal{U}(t) = \sum_{n=0}^{\infty} |\mathcal{U} \cap \mathcal{A}^n| t^n,$$

and $\gamma(\mathcal{U}) = (\limsup_{n\to\infty} |\mathcal{W} \cap \mathcal{A}^n|^{1/n})^{-1}$ is the radius of convergence of $G_\mathcal{U}$. If $\mathcal{U}$ is infinite, the entropy of $\mathcal{U}$ is defined by

$$h(\mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n} \log |\mathcal{U} \cap \mathcal{A}^n|.$$

The following are consequences of definitions.

Remarks 3.2.2. (1) $G_\mathcal{U}(t)$ is increasing on $(0, \gamma(\mathcal{U}))$.

(2) If $|U| < \infty$, then $G_\mathcal{U}$ is a polynomial, $\gamma(\mathcal{U}) = \infty$ and $\lim_{t\to\infty} G_\mathcal{U}(t) = \infty$.

(3) Since $\mathcal{U} \subseteq \mathcal{U}^*$, $\gamma(\mathcal{U}^*) \leq \gamma(\mathcal{U})$.

(4) Suppose that $\mathcal{U}$ is uniquely decipherable. Then the empty block ϵ does not belong to $\mathcal{U}$, so that $G_{\mathcal{U}}(0) = 0$.

(5) If $|\mathcal{U}| = \infty$, then $h(\mathcal{U}) = \log(1/\gamma(\mathcal{U}))$ and $h(\mathcal{U}) \leq h(\mathcal{U}^*)$.

For each $m \geq 1$, let

$$\alpha = \langle \alpha_0, \alpha_1, \cdots, \alpha_{m-1} \rangle \in \mathbb{N}^m,$$
$$\mathcal{U}^*(\alpha) = (\mathcal{U} \cap \mathcal{A}^{\alpha_0}) \cdots (\mathcal{U} \cap \mathcal{A}^{\alpha_{m-1}}) = \prod_{j \in \mathrm{dom}(\alpha)} (\mathcal{U} \cap \mathcal{A}^{\alpha_j}),$$
$$\|\alpha\| = \sum_{j \in \mathrm{dom}(\alpha)} \alpha_j, \quad \|0\| = 0, \quad \text{and} \quad \mathcal{U}^*(0) = \{\epsilon\}.$$

If $w \in \mathcal{U}^*(\alpha)$, then $|w| = \|\alpha\|$. Then $\{\mathcal{U}^*(\alpha) : \alpha \in \bigcup_{n=0}^{\infty} \mathbb{N}^n\}$ is a partition of $\mathcal{U}^*$ when $\mathcal{U}$ is uniquely decipherable. Let

$$\mathbb{N}^*(l) = \{\alpha \in \bigcup_{n=0}^{\infty} \mathbb{N}^n : \|\alpha\| = l\},$$

then each $\{\mathbb{N}^m : m \in \mathbb{N}\}$ and $\{\mathbb{N}^*(l) : l \in \mathbb{N}\}$ is a partition of $\bigcup_{n=0}^{\infty} \mathbb{N}^n$.

$$\left(\sum_{n \in \mathbb{N}} a_n t^n \right)^m = \sum_{\alpha \in \mathbb{N}^m} \left(\prod_{j \in \mathrm{dom}(\alpha)} a_{\alpha_j} \right) t^{\|\alpha\|} \qquad (m \in \mathbb{N}, a_n \geq 0). \tag{3.2}$$

Theorem 3.2.3. *If $\mathcal{U}$ is uniquely decipherable and infinite then*

$$G_{\mathcal{U}^*}(t) = \frac{1}{1 - G_{\mathcal{U}}(t)} \qquad (|t| < r)$$

where $0 < r \leq \gamma(\mathcal{U})$ and $G_{\mathcal{U}}(r) < 1$.

Proof. Suppose that $\mathcal{U}$ is infinite and uniquely decipherable. Let $0 \leq t < r < \gamma(\mathcal{U})$ and $G_{\mathcal{U}}(t) < 1$. It follows from the partitions of $\bigcup_{n=0}^{\infty} \mathbb{N}^n$ and the equation (3.2) that

$$\begin{aligned}\frac{1}{1-G_{\mathcal{U}}(t)} &= \sum_{m\in\mathbb{N}}\left(\sum_{n\in\mathbb{N}}|\mathcal{U}\cap\mathcal{A}^n|t^n\right)^m = \sum_{m\in\mathbb{N}}\sum_{\alpha\in\mathbb{N}^m}\prod_{j\in\mathrm{dom}(\alpha)}|\mathcal{U}\cap\mathcal{A}^{\alpha_j}|t^{\|\alpha\|}\\ &= \sum_{m\in\mathbb{N}}\sum_{\alpha\in\mathbb{N}^m}|\mathcal{U}^*(\alpha)|t^{\|\alpha\|} = \sum_{l\in\mathbb{N}}\sum_{\alpha\in\mathbb{N}^*(l)}|\mathcal{U}^*(\alpha)|t^l\\ &= \sum_{l\in\mathbb{N}}|\mathcal{U}\cap\mathcal{A}^l|t^l = G_{\mathcal{U}^*}(t).\end{aligned}$$

By the identity theorem for holomorphic functions, we obtain the result. □

We use the following to prove Theorem 3.2.1.

Corollary 3.2.4. *Suppose that $\mathcal{U}$ is uniquely decipherable and $h(\mathcal{U}^*) = \log\lambda$.*

(1) *If $\lim_{t\to\gamma(\mathcal{U})} G_{\mathcal{U}}(t) > 1$, then $G_{\mathcal{U}}(1/\lambda) = 1$.*

(2) *If $\lim_{t\to\gamma(\mathcal{U})} G_{\mathcal{U}}(t) \le 1$ then $h(\mathcal{U}) = h(\mathcal{U}^*)$.*

Proof. (1) If $\lim_{t\to\gamma(\mathcal{U})} G_{\mathcal{U}}(t) > 1$, then there is $r \in (0, \gamma(\mathcal{U}))$ such that $G_{\mathcal{U}}(r) = 1$ and $0 \le G_{\mathcal{U}}(t) < 1$ for $0 \le t < r$. If $|t| < r$, then $|G_{\mathcal{U}}(t)| \le G_{\mathcal{U}}(|t|) < 1$ and $1 - G_{\mathcal{U}}(t) \ne 0$, so that $1/(1 - G_{\mathcal{U}})$ is analytic in $|t| < r$. Also it has a pole at $t = r$. Thus the radius of convergence of $1/(1 - G_{\mathcal{U}})$ is r, and $\gamma(\mathcal{U}^*) = r$ by Theorem 3.2.3. Since $h(\mathcal{U}^*) = \log\lambda$, $r = 1/\lambda$ and we obtain the desired result.

(2) If $\lim_{t\to\gamma(\mathcal{U})} G_{\mathcal{U}}(t) \le 1$, then $G_{\mathcal{U}}(t) < 1$ for $0 \le t < \gamma(\mathcal{U})$. As the above proof of (1) with $r = \gamma(\mathcal{U})$ we obtain $\gamma(\mathcal{U}^*) \ge \gamma(\mathcal{U})$. Thus $\gamma(\mathcal{U}^*) = \gamma(\mathcal{U})$, and $h(\mathcal{U}^*) = h(\mathcal{U})$. □

We are now ready to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. We may assume that X is irreducible and $h(X) = \log\lambda$. By passing to a higher block system, if necessary, we may assume that there is a finitary symbol f for (X, σ_X). We set

$$\begin{aligned}\mathcal{B} &= \{w \in \mathcal{B}(X) : f \text{ does not occur in } w\} \text{ and}\\ \mathcal{W} &= \{fw : w \in \mathcal{B},\ fwf \in \mathcal{B}(X)\}.\end{aligned}$$

Then it is clear that $\mathcal{W}$ is uniquely decipherable, and that $\mathcal{B}(X)$ is the set of subblocks of blocks from $\mathcal{W}^*$. Also, $|\mathcal{B}_n(X)| \geq \lambda^n$ for all n.

Remark 3.2.5. Let X be an irreducible shift space. If X is a shift of finite type, $h(\mathcal{B}) < h(X)$ by the Perron-Frobenius Theorem and Theorem 4.4.7 of [LinM]. If X is sofic, then it is a factor of a shift of finite type, so we have $h(\mathcal{B}) < h(X)$. This does not hold in general shift spaces . For example, we construct a coded system Y in Section 5.2 and 5.3 such that $h(\mathcal{B}) = h(Y)$.

Proposition 3.2.6. *$h(\mathcal{W}^*) = h(X)$.*

Proof. Since $\mathcal{W}^* \subseteq \mathcal{B}(X)$, Then $h(\mathcal{W}^*) \leq \log \lambda$. If $h(\mathcal{W}^*) < \log \lambda$, then there are $c, \mu > 0$ such that

$$\mu < \lambda \text{ and } |\mathcal{B} \cap \mathcal{A}^n|, |\mathcal{W}^* \cap \mathcal{A}^n|, |f\mathcal{B} \cap \mathcal{A}^n| \leq c\mu^n$$

where $f\mathcal{B} = \{fw : w \in \mathcal{B}\}$ and $n = 1, 2, \cdots$.

For $\alpha = \langle \alpha_1, \alpha_2, \alpha_3 \rangle \in \mathbb{N}^3$, we put $E(\alpha) = (\mathcal{B} \cap \mathcal{A}^{\alpha_1})(\mathcal{W}^* \cap \mathcal{A}^{\alpha_2})(f\mathcal{B} \cap \mathcal{A}^{\alpha_3})$ and $\mathbb{N}^3(n) = \{\alpha \in \mathbb{N}^3 : \|\alpha\| = n\}$. Then $\mathcal{B}_{\|\alpha\|}(X) \subseteq E(\alpha)$ for some α so that $\mathcal{B}_n(X) \subseteq \bigcup_{\alpha \in \mathbb{N}^3(n)} E(\alpha)$. Since $|E(\alpha)| \leq c^3 \mu^{\|\alpha\|}$ and $|\mathbb{N}^3(n)| \leq dn^3$ for some d, we have $|\mathcal{B}_n(X)| \leq dc^3 n^2 \mu^n$. Then $\lambda \leq \mu$, it is a contradiction. The proof is done. $\square$

We finish the proof of Theorem 3.2.1. Since $h(\mathcal{W}) \leq h(\mathcal{B})$, Remark 3.2.5 and Proposition 3.2.6 imply that $h(\mathcal{W}) < h(\mathcal{W}^*)$. From Corollary 3.2.4 we get $G_{\mathcal{W}}(\lambda^{-1}) = 1$.

We now construct subshifts of X having finite type. Let $k = 1, 2, \cdots$ and $\mathcal{W}_k = \{fw \in \mathcal{W} : |w| \leq k\}$. Then we get coded systems $X_k = \mathbf{X}(\mathcal{W}_k)$ with code $\mathcal{W}_k$. It is obvious that each X_k is irreducible, $X_k \subseteq X_{k+1}$ and $X_k \subseteq X$ for $k = 1, 2, \cdots$. Furthermore each X_k is a shift of finite type. Indeed, f is a finitary symbol for (X_k, σ_{X_k}) by definition of $\mathcal{W}_k$. Since each block $u \in \mathcal{B}_{k+3}(X_k)$ contains f, every block in $\mathcal{B}_{k+3}(X_k)$ is finitary and X_k is a $(k+3)$-step shift of finite type.

Let k be a number such that $h(X_k) = \log \lambda_k > 0$. Since $\mathcal{W}_n \subseteq \mathcal{W}_{n+1}$ and $\mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n$, such a number k exists, and $h(X_n) = \log \lambda_n > 0$ for $n \geq k$. It is obvious that $\mathcal{W}_k$ is the set of blocks fw such that $w \in \mathcal{B}(X_k)$ does not contain the symbol f; hence $\mathcal{B}(X_k)$ is the set of subblocks of a block in $(\mathcal{W}_k)^*$. From Proposition 3.2.6, we have $h((\mathcal{W}_k)^*) = \log \lambda_k$. Since $\mathcal{W}_k$ is finite and uniquely decipherable, Remark 3.2.2 and Corollary 3.2.4 imply that $G_{\mathcal{W}_k}(\lambda_k^{-1}) = 1$.

Since $\{G_{\mathcal{W}}\}$ converges uniformly to $G_{\mathcal{W}}$ on any interval including λ, and since $G_{\mathcal{W}}(\lambda^{-1}) = 1 = G_{\mathcal{W}_k}(\lambda_k^{-1})$, it follows that $\{\lambda_k^{-1}\}$ converges to λ^{-1}. Thus $\lim_{k\to\infty} h(X_k) = \lim_{k\to\infty} \log \lambda_k = \log \lambda = h(X)$, and then X is almost sofic (cf. [Mar, Proposition 3]). $\square$

We conclude this section by examples. An irreducible sofic shift is both coded and almost sofic (Theorem 3.1.4, 3.2.1). There is an irreducible non-sofic shift such that it is both coded and almost sofic.

Recall that a shift space X is (topologically) mixing if, for any two blocks $u, v \in \mathcal{B}(X)$, there is an $N \geq 1$ such that for each $n \geq N$ there is a block $w \in \mathcal{B}(X)$ of length n in X so that $uwv \in \mathcal{B}(X)$ [LinM]. If N is the same value for every ordered pair u, v, then X is said to have the specification property: there is an $N \geq 1$ such that for every ordered pair u, v of blocks in $\mathcal{B}(X)$ and for each $n \geq N$ there is a block $w \in \mathcal{B}(X)$ of length n in X so that $uwv \in \mathcal{B}(X)$ [Ber, Kwa]. It is clear that if a shift space has the specification property then the shift space is irreducible. There are some results of a shift having the specification property [Ber, Buz, Kwa, Spa].

Example 3.2.7. Let Y be a shift space which has the specification property. We claim that Y is coded and almost sofic.

The specification property guarantees the existence of a synchronizing word: $w \in \mathcal{B}(Y)$ is synchronizing if, whenever $uw, u'w \in \mathcal{B}(Y)$, then $uwv \in \mathcal{B}(Y)$ if and only if $u'wv \in \mathcal{B}(Y)$ for $v \in \mathcal{B}(Y)$ [Ber]. It is easy to check that a synchronizing word is finitary, so that X is coded by Theorem 3.1.2. In [Spa, Lemma 6.8 and 6.10], it is shown that Y is almost sofic.

Example 3.2.8. Let Z be the context free shift over $\{a, b, c\}$. Then $Z = \mathbf{X}_{\mathcal{F}}$ where $\mathcal{F} = \{ab^m c^n a : m \neq n\}$, and it is not sofic [LinM, Example 3.1.7]. However, it is coded because

$$\big\{au : u \in \{b,c\}^* \setminus \{b^m c^n : m \neq n, \quad m, n \geq 0\}\big\}$$

is a code for Z. In fact, the symbol a is finitary: let $wa, au \in \mathcal{B}(Z)$. If $v \in \mathcal{F}$ occurs in wau, then it occurs in wa or au, it is a contradiction. Hence $wau \in \mathcal{B}(Z)$ and a is finitary.

The shift Z has the specification property. For every pair of blocks u, v and for each $n \geq 2$ there is a block w of length n such that $uwv \in \mathcal{B}(Z)$. As the above example Z is almost sofic.

There are examples of shift spaces which are not almost sofic.

Example 3.2.9. (1) In [Pet, Section 3, 5] there are shift spaces that are not almost sofic. We, among other things, are interested in the disk system in Section 5 of [Pet]. The shift is neither coded nor almost sofic (Chapter 4).

(2) If an irreducible shift space has no periodic points, it can not contain any subshifts of finite type, so that it is not almost sofic (Theorem 5.2.5). Theorem 5.2.1 shows that there is coded system which is not almost sofic.

Chapter 4

Disk systems and their analogies

Petersen [Pet] introduced a disk system which is periodic saturated, but not almost sofic. Let $c > 0$ and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. For $x \in \{-1, 1\}^{\mathbb{Z}}$ and $m \leq n$, let

$$s_{m,n}(x) = \sum_{k=m}^{n} x_k \alpha^k.$$

We define a labeled graph $\mathcal{G} = (G, \mathcal{L})$ where its vertex set $\mathcal{V} = \mathcal{V}(G) = \{s_{m,n}(x) : x \in \{-1, 1\}^{\mathbb{Z}},\ m \leq n\}$ and the labeling $\mathcal{L}$ by an alphabet {-1, 1}. Let $s \in \mathcal{V}$. Then $s = s_{m,n}(x)$ for some x and $m \leq n$. We draw an edge labeled $a = \pm 1$ from s to $s + a\alpha^{n+1}$.

Let G' be the subgraph of the graph G with vertex set $\mathcal{V}_c = \{s \in \mathcal{V} : |s| < c\}$. Recall that $\widehat{\mathbf{X}}_{G'}$ is the vertex shift over $\mathcal{V}_c$. We define $(\mathcal{V}_c)'$ is the subset of $\mathcal{V}_c$ such that $s \in (\mathcal{V}_c)'$ if and only if for $\epsilon > 0$ there are a point $y \in \widehat{\mathbf{X}}_{G'}$ and a positive integer k with $y_0 = s$ and $|y_k| < \epsilon$. Let H be the subgraph of G' with vertex set $(\mathcal{V}_c)'$. Then H is a subgraph of G, and it induces the labeled subgraph $\mathcal{H}$ of $\mathcal{G}$. The shift space $\mathbf{X}_{\mathcal{H}}$ is called a *disk system* which is a subshift of $\{-1, 1\}^{\mathbb{Z}}$.

The definition of $\mathcal{G}$ guarantees a right infinite sequence ξ on that labeled graph, but ξ can not be extended to the left infinitely: if x is a bi-infinite sequence with $x_{[0,\infty)} = \xi$, then $x_{(-\infty,0]}$ must meet a vertex -1 or 1. Since there are no labeled edges on $\mathcal{G}$ ending at -1 or 1, x is not a bi-infinite sequence on

$\mathcal{G}$.

We can modify the construction in [Pet] to obtain a disk system which satisfies the results in [Pet]. We also show that the disk system has additional properties.

Theorem 4.0.1. *There is a disk system such that*

(1) *it is irreducible,*

(2) *it is periodic points dense,*

(3) *it has positive entropy,*

(4) *it is periodic saturated,*

(5) *it is not almost sofic,*

(6) *it is not a coded system and*

(7) *it has a flip.*

In Section 4.1, we construct a disk system with property (3), (4), (5) and (6). The remaining properties of Theorem 4.0.1 is proved in Section 4.2.

4.1 Construction of a disk system

Let $c > 0$. Suppose from now that $\alpha \in \mathbb{C}$, $|\alpha| = 1$, and $\alpha^n \neq 1$ for all $n = 1, 2, \cdots$. We define some notations. For $w = w_0 \cdots w_{k-1} \in \{-1, 1\}^k$, let

$$\begin{aligned} s_n(w) &= \sum_{m=0}^{n-1} w_m \alpha^m \ (1 \leq n \leq k), \quad s_0(w) = 0, \\ s(w) &= s_k(w) \quad \text{and} \\ r(w) &= \max\{|s_n(w)| : 0 \leq n \leq k\}. \end{aligned}$$

Similarly, we can define $s_n(x)$ and $r(x)$ for $x \in \{-1, 1\}^{\mathbb{N}}$: $s_n(x) = s(x_{[0,n-1]})$ and $r(x) = \sup\{|s_n(x)| : n = 1, 2, \cdots\}$. A right-infinite sequence $x \in \{-1, 1\}^{\mathbb{N}}$

is said to be *recurrent* if there is an infinite sequence $n_1 < n_2 < \cdots$ such that $\lim_{n\to\infty} s_{n_j}(x) = 0$.

The next result shows that every periodic point is recurrent.

Lemma 4.1.1. *Let $x \in \{-1,1\}^{\mathbb{N}}$. If $x = (x_{[0,m-1]})^{\infty}$ for some m, then*

$$s_{mq+r}(x) = \frac{1-\alpha^{mq}}{1-\alpha^{m}} s_m(x) + \alpha^{mq} s_r(x)$$

where $q, r \in \mathbb{N}$ and $0 \le r \le m-1$. Furthermore, every periodic point is recurrent.

Proof. Suppose that $x = (x_{[0,m-1]})^{\infty} \in \{-1,1\}^{\mathbb{N}}$ for some m. It is obvious that the first statement holds. We only show that x is recurrent. Let $\epsilon > 0$. Since $\alpha^n \neq 1$ for all n, there are an infinite sequence $\langle q_j \rangle$ and a positive integer J such that $q_1 < q_2 < \cdots$ and

$$|1-\alpha^{mq_j}| < \frac{|1-\alpha^{m}|}{|s_m(x)|}\epsilon \qquad (j \ge J).$$

Letting $n_j = q_j m$ and $j \ge J$,

$$|s_{n_j}(x)| = |s_{mq_j}(x)| = \frac{|1-\alpha^{mq_j}|}{|1-\alpha^{m}|}|s_m(x)| < \epsilon.$$

Thus x is recurrent. □

Lemma 4.1.2. *Let $\delta > 0$. There is an integer n such that $\alpha^{2n+1} \in \{\exp(i\theta) : |\theta - \pi| < \pi/6\}$ and $|1+\alpha^{2n+1}|/|1+\alpha| < \delta$.*

Proof. Let $\delta > 0$. By the assumptions of α, $\{\alpha^{2k+1} : k \in \mathbb{N}\}$ is dense in the unit circle, so that there is an integer n such that

$$|\alpha^{2n+1} - (-1)| < \min\{2\sin(\pi/12),\ \delta|1+\alpha|\}.$$

Then $|1+\alpha^{2n+1}|/|1+\alpha| < \delta$. Let $\alpha^{2n+1} = \exp(i\theta_1)$ for some θ_1. Since $|\alpha^{2n+1} - (-1)|$ is smaller than $2\sin(\pi/12)$, we obtain $|\theta_1 - \pi| < \pi/6$. □

Let $W(c,\alpha)$ be the set of $x \in \{-1,1\}^{\mathbb{N}}$ such that $r(x) < c$ and x is recurrent. We denote by $\mathcal{B}(c,\alpha)$ the set of all blocks appearing in points of $W(c,\alpha)$, i.e.,

$$\mathcal{B}(c,\alpha) = \{x_{[n,m]} : x \in W(c,\alpha),\ n \le m\}.$$

We will show that the set $\mathcal{B}(c,\alpha)$ is the language of a subshift of $\{-1,1\}^{\mathbb{Z}}$. We will call the shift space a disk system, and denote it by $X(c,\alpha)$. The rest of this section is devoted to find the additional conditions for c and α such that $X(c,\alpha)$ satisfies the properties (3), (4), (5) and (6) of Theorem 4.0.1. More precisely, we prove the following theorems:

Theorem 4.1.3. *If $c > 2/|1+\alpha|$, there is a subshift $X(c,\alpha)$ of $\{-1,1\}^{\mathbb{Z}}$ such that its language is the set $\mathcal{B}(c,\alpha)$. In addition, if $c > \sqrt{2}+3$, then $X(c,\alpha)$ has positive entropy.*

Theorem 4.1.4. *If $c > \max\{2/|1+\alpha|,\ \sqrt{2}+3\}$ and $\epsilon > 0$ and*

$$\lim_{\epsilon \to 0^+} h(X(c-\epsilon,\alpha)) = h(X(c,\alpha)),$$

then $X(c,\alpha)$ is periodic saturated.

A number β is transcendental over $\mathbb{Z}$ if $\sum_{k=0}^{n} a_k \beta^k \neq 0$ for all $a_k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Theorem 4.1.5. *If $c > \max\{2/|1+\alpha|,\ \sqrt{2}+3\}$ and α is transcendental over $\mathbb{Z}$, then $X(c,\alpha)$ does not contain a shift of finite type with positive entropy.*

Corollary 4.1.6. *Let $c > \max\{2/|1+\alpha|,\ \sqrt{2}+3\}$ and let α be transcendental over $\mathbb{Z}$. Then $X(c,\alpha)$ is not almost sofic, and also not coded.*

Proof. The first statement is an immediate consequence of Theorem 4.1.5. For the second statement, suppose that $X(c,\alpha)$ is coded. Then there is a set $\mathcal{C}$ such that $X(c,\alpha) = \mathbf{X}(\mathcal{C})$ and $\mathcal{C}$ is uniquely decipherable (Theorem 3.1.2). From Theorem 4.1.3, $X(c,\alpha)$ has positive entropy, so there are two different blocks $u, v \in \mathcal{C}$ such that $\mathbf{X}(\{u,v\})$ is a subshift of $\mathbf{X}(\mathcal{C})$. Since $\mathbf{X}(\{u,v\})$ is a shift of finite type with positive entropy, it contradicts to Theorem 4.1.5. $\square$

In the rest we will prove Theorem 4.1.3, 4.1.4 and 4.1.5. To prove these theorems we need some lemmas. Lemma 4.1.1 implies that a periodic point is a candidate for a point in $W(c,\alpha)$. In fact, if c is sufficiently large, every block in $\mathcal{B}(c,\alpha)$ occurs in a periodic point in $W(c,\alpha)$.

Lemma 4.1.7. *Let $c > 2/|1+\alpha|$. If $x \in W(c,\alpha)$ and $n \in \mathbb{N}$, then there is a point $x' \in W(c,\alpha)$ such that x' is periodic and contains $x_{[0,n-1]}$.*

Proof. The proof is important as the result. We use the following technique for proving lemmas and theorems in the rest of this section.

Let $c > 2/|1+\alpha|$. Suppose that $x \in W(c,\alpha)$ and $n \in \mathbb{N}$. Choose a positive number

$$\delta = \frac{1}{6}\Big(c - \max\{r(x),\, 2/|1+\alpha|\}\Big).$$

We may assume that $|s_n(x)| < \delta$ (since x is recurrent, there is a positive number $m \geq n$ with $|s_m(x)| < \delta$).

We need consider two cases: first, we assume that $|1-\alpha^n| \geq \sqrt{2}$. Define $x' \in \{-1,1\}^{\mathbb{N}}$ to be a sequence with

$$x' = (x_{[0,n-1]})^{\infty}.$$

It is clear x' is periodic and contains $x_{[0,n-1]}$. It remains to show that $x' \in W(c,\alpha)$. Lemma 4.1.1 implies that x' is recurrent and

$$\begin{aligned} |s_{nq+r}(x')| &\leq \frac{|1-\alpha^{nq}|}{|1-\alpha^n|}|s_n(x')| + |\alpha^{nq}||s_r(x')| \\ &= \frac{|1-\alpha^{nq}|}{|1-\alpha^n|}|s_n(x)| + |s_r(x)| \leq \sqrt{2}\delta + r(x), \end{aligned} \tag{4.1}$$

the third inequality holds because $|1-\alpha^{nq}| \leq 2$ and $|1-\alpha^n| \geq \sqrt{2}$. Therefore

$$r(x') \leq \sqrt{2}\delta + r(x) < 6\delta + r(x) \leq c,$$

so that $x' \in W(c,\alpha)$.

We now assume that $|1-\alpha^n| < \sqrt{2}$. By Lemma 4.1.2 there is an odd integer m such that $|1+\alpha^m| < 2\sin(\pi/12)$ and $|1+\alpha^m|/|1+\alpha| < \delta$. Let

$x' \in \{-1, 1\}^{\mathbb{N}}$ with $x' = (x'_{[0,n+m-1]})^\infty$ and

$$x'_{[0,n+m-1]} = x_{[0,n-1]}1(-1)1(-1)\cdots 1(-1)1.$$

As the above it is enough to show that $r(x') < c$. Let $q \in \mathbb{N}$ and $0 \leq r \leq n + m - 1$. As the above, from Lemma 4.1.1 we obtain

$$\begin{aligned} |s_{(n+m)q+r}(x')| &\leq \frac{2|s_{n+m}(x')|}{|1 - \alpha^{n+m}|} + |s_r(x')| \\ &< 2|s_{n+m}(x')| + |s_r(x')|, \end{aligned} \tag{4.2}$$

where the second inequality follows from the assumption of α^n and α^m. Indeed, we can write $\alpha^n = \exp(i\theta_1)$ and $\alpha^m = \exp(i\theta_2)$ where $0 \leq \theta_1, \theta_2 \leq 2\pi$. Then, since $|1 - \alpha^n| < \sqrt{2}$ and $|\alpha^m + 1| < 2\sin(\pi/12)$, we obtain $|\theta_1| < \pi/2$ and $|\theta_2| > 5\pi/6$, so that $|\theta_1 + \theta_2| > \pi/3$. Thus $|1 - \alpha^{n+m}| > 1$.

We compute the values $|s_{n+m}(x')|$ and $|s_r(x')|$: for $0 \leq k \leq n$

$$s_k(x') = s_k(x),$$

and for $n + 1 \leq k \leq n + m$

$$\begin{aligned} s_k(x') &= s_n(x) + \alpha^n - \alpha^{n+1} + \cdots + (-1)^{k-n-1}\alpha^{k-1} \\ &= s_n(x) + \frac{\alpha^n + (-1)^{k-n-1}\alpha^k}{1 + \alpha}. \end{aligned}$$

Thus we obtain, if $k = n + m$ then

$$\begin{aligned} |s_{n+m}(x')| &\leq |s_n(x)| + \frac{|\alpha^n||1 + (-1)^{m-1}\alpha^m|}{|1 + \alpha|} \\ &= |s_n(x)| + \frac{|1 + \alpha^m|}{|1 + \alpha|} < 2\delta, \end{aligned} \tag{4.3}$$

and if $k = r$ then

$$|s_r(x')| \leq \begin{cases} |s_r(x)| & (0 \leq r \leq n) \\ |s_n(x)| + 2/|1+\alpha| & (n+1 \leq r \leq n+m-1) \end{cases}$$
$$\leq \delta + \max\{r(x),\, 2/|1+\alpha|\} \tag{4.4}$$

since $|\alpha^n + (-1)^{k-n-1}\alpha^k| = |\alpha^n||1+(-1)^{k-n-1}\alpha^{k-n}| \leq 2$.

Equations (4.2), (4.3) and (4.4) show that

$$|s_{(n+m)q+r}(x')| < 5\delta + \max\{r(x),\, 2/|1+\alpha|\},$$

and $r(x') \leq 5\delta + \max\{r(x),\, 2/|1+\alpha|\} < c$. The proof is done. $\square$

Let X be a shift space. It is difficult to calculate directly $|\mathcal{B}_n(X)|$ for $n \in \mathbb{N}$. If we find a lower bound of $|\mathcal{B}_n(X)|$ and the lower bound is strictly larger than 1, then the entropy $h(X)$ is positive.

For $n \in \mathbb{N}$, let

$$\mathcal{B}_n(c,\alpha) = \{w \in \mathcal{B}(c,\alpha) : |w| = n\}.$$

We will define an injective map from $\{-1,1\}^n$ to $\mathcal{B}(c,\alpha)$. It guarantees a lower bound of $\mathcal{B}_m(c,\alpha)$ for $m \in \mathbb{N}$.

Lemma 4.1.8. *Let $0 < \delta < 1$. There are a positive number N and a finite subset F of $\{-1,1\}^{\mathbb{N}}$ such that*

(1) *$r(w) < \sqrt{2}+2$ for $w \in F$ and*

(2) *if $z \in A(\delta)$, then there is a block $w \in F$ with $|s(w) - z| < \delta$,*

where $A(\delta) = \{z : 1-\delta < |z| < 1+\delta\}$.

Proof. Suppose that $0 < \delta < 1$. We denote that $D = \{2(\alpha^i + \alpha^j) : i, j \in 2\mathbb{N}\}$. Let $N(z,r) = \{z' : |z - z'| < r\}$ for $r > 0$.

Since $A(\delta) \subseteq N(0,4) \subseteq \bar{D}$ and $A(\delta) \cap D$ is totally bounded, there is a finite subset I of $2\mathbb{N} \times 2\mathbb{N}$ such that

(i) $i < j$ for $(i,j) \in I$,

(ii) $2(\alpha^i + \alpha^j) \in A(\delta)$ for $(i,j) \in I$, and

(iii) $A(\delta) \subseteq \bigcup_{(i,j)\in I} N\Big(2(\alpha^i + \alpha^j), \delta/2\Big)$.

Let x be a right-infinite sequence in $\{-1,1\}^{\mathbb{N}}$ defined by

$$x = \begin{cases} 1^\infty & \text{if } \sqrt{2} < |1-\alpha| \\ (1(-1))^\infty & \text{if } \sqrt{2} < |1+\alpha|. \end{cases}$$

Choose a number N such that $|s_N(x)| < \delta/2$ and $N > j$ for all $(i,j) \in I$. Let $(i,j) \in I$. We define $w(i,j)$ to be the block obtained from $x_{[0,N-1]}$ by replacing x_t with $-x_t$ for $t = i, j$; for example, if $N = 5$ and $(0,2) \in I$, then $w(0,2) = (-x_0)x_1(-x_2)x_3x_4$.

Let F be the set of such blocks $w(i,j)$ for $(i,j) \in I$. It is clear that F is a finite subset of $\{-1,1\}^N$. Let $w = w(i,j) \in F$. We compute the value $|s_k(w)|$ for $0 \le k \le N$. From the definition of x, the ith symbol and the jth symbol of w are both the symbol -1. Then

$$\begin{aligned} |s_k(w)| &= \begin{cases} |s_k(x)| & (0 \le k \le i) \\ |s_k(x) - 2\alpha^i| & (i+1 \le k \le j) \\ |s_k(x) - 2(\alpha^i + \alpha^j)| & (j+1 \le k \le N) \end{cases} \qquad (4.5) \\ &\le \begin{cases} r(x) & (0 \le k \le i) \\ r(x) + 2 & (i+1 \le k \le j) \\ r(x) + 1 + \delta & (j+1 \le k \le N), \end{cases} \end{aligned}$$

the last inequality holds because of the condition (ii): $|2(\alpha^i + \alpha^j)| < 1 + \delta$. Thus $r(w) \le r(x) + 2 < \sqrt{2} + 2$ and the condition (1) holds. To show that the condition (2) holds, suppose that $z \in A(\delta)$. Then $-z \in A(\delta)$ and so that there is $(i_0, j_0) \in I$ with $-z \in N\Big(2(\alpha^{i_0} + \alpha^{j_0}), \delta/2\Big)$. Letting $w = w(i_0, j_0) \in F$, we obtain

$$\begin{aligned} |s(w)-z| &= |s_N(w)-z| = |s_N(x) - 2(\alpha^{i_0}+\alpha^{j_0}) - z| \\ &\le |s_N(x)| + |2(\alpha^i+\alpha^j) - (-z)| < \delta \end{aligned}$$

from equation (4.5). The proof is done. □

Using the result of Lemma 4.1.8 we define a map Φ_n on $\{-1,1\}^n$ for each $n \in \mathbb{N}$.

Corollary 4.1.9. *Let δ, N and F be given as in Lemma 4.1.8. For each $n \in \mathbb{N}$ and each block $a_0 \cdots a_{n-1} \in \{-1,1\}^n$, there are n blocks $w_0, \cdots, w_{n-1}$ in F such that*

(1) $|s(a_0 w_0 \cdots a_{n-1} w_{n-1})| < \delta$ *and*

(2) $r(a_0 w_0 \cdots a_{n-1} w_{n-1}) < \sqrt{2} + 3 + \delta$.

Moreover, if $c > \max\{\sqrt{2}+3,\ 2/|1+\alpha|\}$, then the map Φ_n defined by

$$\Phi_n(a_0 \cdots a_{n-1}) = a_0 w_0 \cdots a_{n-1} w_{n-1}$$

is an injection map from $\{-1,1\}^n$ to $\mathcal{B}_{n(N+1)}(c,\alpha)$.

Proof. Suppose that δ, N and F be given as in Lemma 4.1.8. We use the induction on $n \in \mathbb{N}$. Let $n = 1$ and $a \in \{-1,1\}$. Since $|-a\alpha^{-1}| \in (1-\delta, 1+\delta)$, Lemma 4.1.8 implies that there is a block w in F with and $r(w) < \sqrt{2}+2$ and $|s(w) - (-a\alpha^{-1})| < \delta$. Then

$$|s(aw)| = |a + \alpha s(w)| = |s(w) - (-a\alpha^{-1})| < \delta$$

since $|\alpha| = 1$. Also, $r(aw) \le 1 + r(w) < \sqrt{2} + 3$.

Suppose, for a block $a_0 \cdots a_{n-1} \in \{-1,1\}^n$, that there are n blocks $w_0, \cdots,$ w_{n-1} in F such that $v := a_0 w_0 \cdots a_{n-1} w_{n-1}$ satisfies the conditions (1) and (2). Let $b \in \{-1,1\}$. We claim that there is a block $u \in F$ such that $vbu = a_0 w_0 \cdots a_{n-1} w_{n-1} bu$ satisfies the condition (1) and (2). We put

$$z = -\alpha^{-n(N+1)-1} s(v) - b\alpha^{-1}.$$

Since $|s(v)| < \delta$ and $|b\alpha^{-1}| = 1$, we obtain $1-\delta < |z| < 1+\delta$. From Lemma 4.1.8 we can find a block $u \in F$ such that $r(u) < \sqrt{2}+2$ and $|s(u)-z| \le \delta$. Thus

$$\begin{aligned}\delta > |s(u)-z| &= |s(u) + \alpha^{-n(N+1)-1}s(v) + b\alpha^{-1}| \\ &= |s(v) + b\alpha^{n(N+1)} + s(u)\alpha^{n(N+1)+1}| = |s(vbu)|\end{aligned}$$

since $|\alpha^{n(N+1)+1}| = 1$. Now we compute $r(vbu)$.

$$\begin{aligned}|s_k(vbu)| &= \begin{cases} |s_k(v)| & \text{if } 0 \le k \le n(N+1) \\ |s(v) + b\alpha^{n(N+1)}| & \text{if } k = n(N+1)+1 \\ |s(vb) + s(u)\alpha^{n(N+1)+1}| & \text{if } n(N+1)+2 \le k \le (n+1)(N+1) \end{cases} \\ &\le \begin{cases} r(v) & \text{if } 0 \le k \le n(N+1) \\ \delta + 1 & \text{if } k = n(N+1)+1 \\ \delta + 1 + r(u) & \text{if } n(N+1)+2 \le k \le (n+1)(N+1) \end{cases}\end{aligned}$$

Since $r(v) < \sqrt{2}+3+\delta$ and $r(u) < \sqrt{2}+2$, we obtain $r(vbu) < \sqrt{2}+3+\delta$.

Finally Suppose that $c > \max\{\sqrt{2}+3,\ 2/|1+\alpha|\}$. Let

$$\delta = \frac{1}{M}\big(c - \max\{\sqrt{2}+3,\ 2/|1+\alpha|\}\big)$$

where $M = \max\{6, c\}$. Then $0 < \delta < 1$. For each n we define a map Φ_n from $\{-1,1\}^n$ to $\mathcal{B}_{n(N+1)}(c,\alpha)$ by

$$\Phi_n(a_0 \cdots a_{n-1}) = a_0 w_0 \cdots a_{n-1} w_{n-1}$$

where $a_0 \cdots a_{n-1} \in \{-1,1\}^n$, and $w_0, \cdots, w_{n-1}$ are given in the above argument with δ and the block $a_0 \cdots a_{n-1}$. Suppose that we proved there is a point y in $W(c,\alpha)$ containing $\Phi_n(a_0 \cdots a_{n-1})$. Then it is clear that Φ_n is well-defined and injective. It remains to define the point y.

Let $m = n(N+1)$. Suppose that $u = \Phi_n(a_0 \cdots a_{n-1})$, then

$$|s(u)| < \delta \quad \text{and} \quad r(u) < \sqrt{2}+3+\delta$$

from the conditions (1) and (2).

As in the proof of Lemma 4.1.7 we consider two cases. First, let $|1-\alpha^m| \geq \sqrt{2}$. Define y to be the right-infinite sequence in $\{-1,1\}^{\mathbb{N}}$ with $y = u^\infty$. It is obvious from Lemma 4.1.1 that y is recurrent. Let $q \in \mathbb{N}$ and $0 \leq r \leq m-1$. Lemma 4.1.1 implies that

$$\begin{aligned}|s_{mq+r}(y)| &\leq \frac{|1-\alpha^{mq}|}{|1-\alpha^m|}|s_m(y)| + |s_r(y)| \leq \frac{2}{\sqrt{2}}|s(u)| + |s_r(u)| \\ &< \sqrt{2}\delta + r(u) < 3\delta + \sqrt{2} + 3.\end{aligned}$$

Thus $r(y) \leq 3\delta + \sqrt{2} + 3 < c$, and $y \in W(c,\alpha)$.

Second, let $|1-\alpha^m| < \sqrt{2}$. Suppose that $m = n(N+1)$. As in the proof of Lemma 4.1.7, we can choose an odd integer k with $|1+\alpha^k| < 2\sin(\pi/12)$, $|1+\alpha^k|/|1+\alpha| < \delta$ and $|1-\alpha^{m+k}| > 1$. Let $y \in \{-1,1\}^{\mathbb{N}}$ be defined by $y = (y_{[0,m+k-1]})^\infty$ with

$$y_{[0,m+k-1]} = u1(-1)1(-1)\cdots 1(-1)1.$$

To show that $y \in W(c,\alpha)$ it is enough to prove that $r(y) < c$. Let $q \in \mathbb{N}$ and $0 \leq r \leq m+k-1$. As in the proof of Lemma 4.1.7

$$|s_{m+k}(y)| \leq |s(u)| + \frac{|1+\alpha^k|}{|1+\alpha|} < 2\delta.$$

Also, if $0 \leq r \leq m$, then

$$|s_r(y)| = |s_r(u)| \leq r(u) < \sqrt{2} + 3 + \delta, \text{ and}$$

if $m+1 \leq r \leq m+k-1$, then

$$|s_r(y)| \leq |s(u)| + \frac{2}{|1+\alpha|} < \delta + \frac{2}{|1+\alpha|},$$

so that $s_r(y) < \delta + \max\{\sqrt{2}+3,\ 2/|1+\alpha|\}$. Therefore

$$\begin{aligned} |s_{(m+k)q+r}(y)| &\le \frac{|1-\alpha^{(m+k)q}|}{|1-\alpha^{m+k}|}|s_{m+k}(y)| + |s_r(y)| \\ &< 4\delta + \delta + \max\{\sqrt{2}+3,\ 2/|1+\alpha|\} \\ &= 5\delta + \max\{\sqrt{2}+3,\ 2/|1+\alpha|\}, \end{aligned}$$

and $r(y) \le 5\delta + \max\{\sqrt{2}+3,\ 2/|1+\alpha|\} < c$, so that $y \in W(c,\alpha)$. The proof is done. $\square$

We are now ready to prove Theorem 4.1.3.

Proof of Theorem 4.1.3. Suppose that $c > \max\{2/|1+\alpha|,\ \sqrt{2}+3\}$. We show that for each $w \in \mathcal{B}(c,\alpha)$ the following hold:

(i) every subblock of w belongs to $\mathcal{B}(c,\alpha)$ and

(ii) there are nonempty block $u, v \in \mathcal{B}(c,\alpha)$ such that $uwv \in \mathcal{B}(c,\alpha)$.

By Proposition 2.1.1(2), $\mathcal{B}(c,\alpha)$ is the language of a shift space, say $X(c,\alpha)$. It is obvious that the first property holds. Lemma 4.1.7 implies that there is a point $(w')^\infty$ in $W(c,\alpha)$ such that w occurs in w'. Thus the second property holds.

We compute the values $\mathcal{B}_n(X(c,\alpha))$ for $n = 1, 2, \cdots$. For each n, from Corollary 4.1.9,

$$2^n = |\Phi_n(\{-1,1\}^n)| \le |\mathcal{B}_{n(N+1)}(c,\alpha)|$$

where Φ_n and N are given in Lemma 4.1.8 and Corollary 4.1.9. Since $\mathcal{B}(c,\alpha) = \mathcal{B}(X(c,\alpha))$, we obtain

$$\frac{1}{n(N+1)} \log |\mathcal{B}_{n(N+1)}(X(c,\alpha))| \ge \frac{1}{N+1} \log 2$$

and $h(X(c,\alpha)) > 0$. $\square$

We denote by $\mathcal{P}(c,\alpha)$ the set of blocks w such that every subblock of w^∞ belongs to $\mathcal{B}(c,\alpha)$.

Lemma 4.1.10. *Let $c > 2/|1+\alpha|$ and $\epsilon > 0$. There is a subset E of $\mathcal{B}(c,\alpha)$ such that*

(1) *E is finite,*

(2) *for all $w \in \mathcal{B}(c,\alpha)$, there are $u, v \in E$ with $uwv \in \mathcal{P}(c+\epsilon,\alpha)$.*

Proof. Let $c > 2/|1+\alpha|$ and $\epsilon > 0$. Suppose that a positive number δ_1 satisfies the following inequality:

$$0 < \delta_1(11 + 18c) < \epsilon.$$

Write

$$S_0 = \{(s_n(x), \alpha^n) : x \in W(c,\alpha),\ n \in \mathbb{N}\}.$$

Since S_0 is bounded in $\mathbb{C}^2$, there is a finite subset S of S_0 such that $S_0 \subseteq \bigcup_{(s_1,s_2)\in S} N(s_1,\delta_1) \times N(s_2,\delta_1)$. For each $(s,t) \in S$ we can choose a point $x \in W(c,\alpha)$, and let W be the set of such points. Then $|W| \le |S|$ and W is a finite subset of $W(c,\alpha)$. Let $R = \max\{r(x) : x \in W\}$,

$$\delta_2 = \frac{1}{6}(c - \max\{R, 2/|1+\alpha|\}) > 0 \quad \text{and} \quad 0 < \delta < \min\{\delta_1, \delta_2\}.$$

Suppose that $x \in W$ and $(s_n(x), \delta^n) \in S$. There is the smallest number $m(x)$ with $m(x) > n$ and $|s_{m(x)}| < \delta$. Let

$$E_1 = \{x_{[0,n-1]} : x \in W,\ (s_n(x), \alpha^n) \in S\},$$
$$E_2 = \{x_{[n,m(x)-1]} : x \in W,\ (s_n(x), \alpha^n) \in S\}.$$

It is clear that E_1 and E_2 are finite subsets of $\mathcal{B}(c,\alpha)$ since W is a finite subset of $W(c,\alpha)$. From Lemma 4.1.2 there is an odd integer d such that $\alpha^d \in \{\exp(i\theta) : |\theta - \pi| < \pi/6\}$ and $|1+\alpha^d|/|1+\alpha| < \delta$. Here ω denotes the block $1(-1)1(-1)\cdots 1(-1)1$ of length d. Let

$$E_3 = \{u\omega : u \in E_2\}.$$

Then E_3 is finite since E_2 is finite. To show that E_3 is a subset of $\mathcal{B}(c,\alpha)$ let

$x \in W$, $(s_n(x), \alpha) \in S$ and $x_{[n,m(x)-1]}\omega \in E_3$. We will define a periodic point $y \in \{-1,1\}^{\mathbb{N}}$ starting with $x_{[0,m(x)-1]}\omega$ and $r(y) < c$. We have two case:(i) $|1-\alpha^l| \geq \sqrt{2}$, (ii) $|1-\alpha^l| < \sqrt{2}$ where $m = m(x)$ and $l = m+d$.

First, suppose that $|1-\alpha^l| \geq \sqrt{2}$. Define y to be $(y_{[0,l-1]})^\infty$ with $y_{[0,l-1]} = x_{[0,m-1]}\omega$. If $0 \leq k \leq m$, then $|s_k(y)| = |s_k(x)| \leq r(x)$. Let $k = m+t$ where $t = 1, 2, \cdots, d$. Then

$$\begin{aligned} s_k(y) &= s_m(x) + \alpha^m(1 - \alpha + \cdots + (-1)^{t-1}\alpha^{t-1}) \\ &= s_m(x) + \alpha^m \frac{1+(-1)^{t-1}\alpha^t}{1+\alpha} \end{aligned}$$

and so that

$$|s_k(y)| \leq |s_m(x)| + \frac{|1+(-1)^{t-1}\alpha^t|}{|1+\alpha|}.$$

Lemma 4.1.1 implies that, for $q \in \mathbb{N}$ and $0 \leq r \leq d-1$,

$$\begin{aligned} |s_{lq+r}(y)| &\leq \frac{|1-\alpha^{lq}|}{|1-\alpha^l|}|s_l(y)| + |s_r(y)| \\ &\leq 2\sqrt{2}\delta + \max\{r(x), \delta + (2/|1+\alpha|)\} \end{aligned}$$

since $|1-\alpha^l| \geq \sqrt{2}$. Therefore, since $\delta < \delta_2$,

$$\begin{aligned} r(y) &< 2\sqrt{2}\delta_2 + \max\{r(x), \delta_2 + (2/|1+\alpha|)\} \\ &< 6\delta_2 + \max\{r(x), \delta_2 + (2/|1+\alpha|)\}. \end{aligned} \tag{4.6}$$

If $r(x) \geq \delta_2 + (2/|1+\alpha|)$, then $R \geq r(x) \geq 2/|1+\alpha|$ and $\delta_2 = (c-R)/2$. Thus the inequality (4.6) is smaller than c. If $r(x) < \delta_2 + (2/|1+\alpha|)$, since $\max\{R, 2/|1+\alpha|\} \geq 2/|1+\alpha|$, the inequality (4.6) is smaller than c. Thus $y \in W(c,\alpha)$.

Suppose that $|1-\alpha^l| < \sqrt{2}$. From Lemma 4.1.2, there is an odd integer d' such that $d' > d$, $\alpha^{d'} \in \{\exp(i\theta) : |\theta - \pi| < \pi/6\}$ and $|1+\alpha^{d'}|/|1+\alpha| < \delta$. Let $\omega' = 1(-1)\cdots 1(-1)1$ and $|\omega'| = d'$. In fact, ω is a prefix of ω'. Thus letting $l' = l + d'$ and $y = (y_{[0,l'-1]})^\infty$ with $y_{[0,l'-1]} = x_{[0,m-1]}\omega'$, we obtain $|1-\alpha^{l'}| > 1$

and the similar result of the above argument

$$|s_{l'q+r}(y)| \le 4\delta_2 + \max\{r(x) + \delta_2 + (2/|1+\alpha|)\},$$

so that $r(y) < 6\delta_2 + \max\{r(x) + \delta_2 + (2/|1+\alpha|)\} < c$, and $y \in W(c,\alpha)$. In either case $x_{[n,m(x)-1]}\omega = y_{[n,m+d-1]} \in \mathcal{B}(c,\alpha)$.

Let $E = E_1 \cup E_2 \cup E_3$. Finally we show that the second statement holds. Let $w \in \mathcal{B}(c,\alpha)$. There are $x \in W(c,\alpha)$, $y \in W$ and $m, n, p, q \in \mathbb{N}$ such that

(i) $x_{[m,m+n-1]} = w$,

(ii) $|s_p(y) - s_m(x)| < \delta_1$, $|\alpha^p - \alpha^m| < \delta_1$ and

(iii) $|s_q(z) - s_{m+n}(x)| < \delta_1$, $|\alpha^q - \alpha^{m+n}| < \delta_1$

from the assumptions of S_0, S, W. Let $u = y_{[0,p-1]}$ and

$$v = \begin{cases} z_{[q,m(z)-1]} & \text{if } \alpha^l \in \{\exp(i\theta) : |\theta - \pi| < \pi/2\} \\ z_{[q,m(z)-1]}\omega & \text{if } \alpha^l \in \{\exp(i\theta) : |\theta| < \pi/2\} \end{cases}$$

where $l = p+n+m(z)-q$. To prove that $uwv \in \mathcal{P}(c+\epsilon,\alpha)$ it is enough to show that $r((uwv)^\infty) < c+\epsilon$. Since a periodic point is recurrent, if $r((uwv)^\infty) < c+\epsilon$ then $(uwv)^\infty \in W(c,\alpha)$. Then every subblock of $(uwv)^\infty$ occur in $\mathcal{B}(c,\alpha)$ and we obtain the desired result.

We have $|1-\alpha^l| \ge \sqrt{2}$ if $\alpha^l = \exp(i\theta)$ for $|\theta - \pi| < \pi/2$, and $|1-\alpha^l| < \sqrt{2}$ otherwise. We can apply the method in the proof in Lemma 4.1.7 to show that $r((uwv)^\infty) < c+\epsilon$. Nevertheless, we present the proof here for the reader's convenience. Let $x' = (uwv)^\infty$ and $|uwv| = L$. Then $L = l$ if $v = z_{[q,m(z)-1]}$, and $L = l+d$ otherwise. We divide the proof into four parts:

[Part I] If $0 \le k \le p$ then $|s_k(x')| = |s_k(y)| \le r(y)$.

[Part II] Let $k = p+t$ where $t = 1, \cdots, n$. Then

$$\begin{aligned} s_k(x') &= s_p(y) + \alpha^p x_m + \cdots + \alpha^{p+t-1} x_{m+t-1} \\ &= s_p(y) + \alpha^p x_m + \cdots + \alpha^{p+t-1} x_{m+t-1} + s_{m+t}(x) - s_{m+t}(x) \\ &= (s_p(y) - s_m(x)) + (\alpha^p - \alpha^m)\{\alpha^{-m}(s_{m+t}(x) - s_m(x))\} + s_{m+t}(x) \end{aligned}$$

since

$$s_{m+t}(x) = s_m(x) + \alpha^m(x_m + \cdots + \alpha^{t-1}x_{m+t-1}).$$

We obtain $|s_k(x')| \leq \delta_1 + 2\delta_1 c + c$ from the condition (ii).

[Part III] Let $k = p + n + t$ where $t = 1, \cdots, m(z) - q$. As [Part II] we obtain

$$\begin{aligned}
s_k(x') &= s_{p+n}(x') + \alpha^{p+n}(z_q + \cdots + \alpha^{t-1}z_{q+t-1}) \\
&= s_{p+n}(x') + \alpha^{p+n}(z_q + \cdots + \alpha^{t-1}z_{q+t-1}) + s_{q+t}(z) - s_{q+t}(z) \\
&= s_{p+n}(x') + \alpha^{p+n}\alpha^{-q}(s_{q+t}(z) - s_q(z)) + s_{q+t}(z) \\
&\quad - s_q(z) - \alpha^q\alpha^{-q}(s_{q+t}(z) - s_q(z)) \\
&= (s_p(y) - s_m(x)) + (\alpha^p - \alpha^m)\{\alpha^{-m}(s_{m+n}(x) - s_m(x))\} + s_{m+n}(x) \\
&\quad + (\alpha^{p+n} - \alpha^q)\alpha^{-q}(s_{q+t}(z) - s_q(z)) - s_q(z) + s_{q+t}(z) \\
&= (s_p(y) - s_m(x)) + (\alpha^p - \alpha^m)\{\alpha^{-m}(s_{m+n}(x) - s_m(x))\} \\
&\quad + \{\alpha^n(\alpha^p - \alpha^m) + (\alpha^{m+n} - \alpha^q)\}\alpha^{-q}(s_{q+t}(z) - s_q(z)) \\
&\quad + (s_{m+n}(x) - s_q(z)) + s_{q+t}(z).
\end{aligned}$$

From the conditions (ii) and (iii) we obtain

$$|s_k(x')| \leq 2\delta_1 + 6\delta_1 c + c \qquad (t = 1, \cdots, m(z) - q - 1)$$

If $t = m(z) - q$, then $k = l$ and

$$|s_l(x')| \leq 2\delta_1 + 6\delta_1 c + \delta < 3\delta_1 + 6\delta_1 c.$$

[Part IV] Finally we calculate $s_k(x')$ for $k = l + 1, \cdots, l + d$ in the case when $v = z_{[q,m(z)-1]}\omega$. Let $k = l + t$ where $t = 1, \cdots, d$. Then

$$\begin{aligned}
s_k(x') &= s_l(x') + \alpha^{l+1}(1 - \alpha + \alpha^2 + \cdots + (-1)^{t-1}\alpha^{t-1}) \\
&= s_l(x') + \alpha^{l+1}\frac{1 + (-1)^{t-1}\alpha^t}{1 + \alpha}.
\end{aligned}$$

From the result of [Part III] we obtain

$$|s_k(x')| \leq 3\delta_1 + 6\delta_1 c + \frac{2}{|1+\alpha|} \qquad (t = 1, \cdots, d-1).$$

If $t = d$, then $k = l + d$ and from the assumption of d

$$|s_{l+d}(x')| \leq 3\delta_1 + 6\delta_1 c + \delta < 4\delta_1 + 6\delta_1 c.$$

If $v = z_{[q,m(z)-1]}$, then $|1 - \alpha^L| = |1 - \alpha^l| \geq \sqrt{2}$. If $v = z_{[q,m(z)-1]}\omega$, then $|1 - \alpha^L| = |1 - \alpha^{l+d}| > 1$ since $\alpha^d = \exp(i\theta)$ for $|\theta - \pi| < \pi/6$. In either case $|1 - \alpha^L| > 1$. Also,

$$|s_L(x')| \leq 3\delta_1 + 6\delta_1 c, 4\delta_1 + 6\delta_1 c \leq 4\delta_1 + 6\delta_1 c \quad \text{and}$$
$$|s_r(x')| \leq 2\delta_1 + 6\delta_1 c + c, 3\delta_1 + 6\delta_1 c + c \leq 3\delta_1 + 6\delta_1 c + c$$

where $L = l, l + d$ and $0 \leq r \leq L - 1$. In either case, Lemma 4.1.1 implies that

$$\begin{aligned} |s_{tL+r}(x')| &\leq 2|s_L(x')| + |s_r(x')| < 2(4\delta_1 + 6\delta_1 c) + 3\delta_1 + 6\delta_1 c + c \\ &= c + \delta_1(11 + 18c) \end{aligned}$$

where $t \in \mathbb{N}$ and $0 \leq r \leq L - 1$. Thus $r(x') \leq c + \delta_1(11 + 18c) < c + \epsilon$. The proof is done. $\square$

We are now ready to prove Theorem 4.1.4.

Proof of Theorem 4.1.4. Suppose that $c > \max\{2/|1 + \alpha|, \sqrt{2} + 3\}$ and $\epsilon > 0$ and $\lim_{\epsilon \to 0^+} h(X(c - \epsilon, \alpha)) = h(X(c, \alpha))$. For $n \in \mathbb{N}$, let $\mathcal{P}_n(c, \alpha)$ be the set of blocks $w \in \mathcal{P}(c, \alpha)$ with $|w| = n$. Since $\mathcal{P}_n(c, \alpha) \subseteq \mathcal{B}_n(X(c, \alpha))$, it is enough to show that

$$\limsup_{n \to \infty} \frac{1}{n} \log |\mathcal{P}_n(c, \alpha)| \geq h(X(c, \alpha)).$$

Let E be the finite subset of $\mathcal{B}(c, \alpha)$ with the property in Lemma 4.1.10. For each $w \in \mathcal{B}(c, \alpha)$ we choose a pair $(u, v) \in E \times E$ and put $f(w) = uwv$. Since E is finite, there is a finite number $N = 2\max\{|u| : u \in E\}$. Let $n \in \mathbb{N}$ and

$w \in \mathcal{B}_n(c, \alpha)$. Since

$$f(w) \in \mathcal{P}_n(c+\epsilon, \alpha) \cup \mathcal{P}_{n+1}(c+\epsilon, \alpha) \cup \cdots \cup \mathcal{P}_{n+N}(c+\epsilon, \alpha) \text{and}$$

$$|f^{-1}(\mathcal{P}_{n+k}(c+\epsilon, \alpha))| \le (k+1)|\mathcal{P}_{n+k}(c+\epsilon, \alpha)| \qquad (0 \le k \le N)$$

we obtain

$$\begin{aligned}|\mathcal{B}_n(c, \alpha)| &= \sum_{k=0}^{N} |f^{-1}(\mathcal{P}_{n+k}(c+\epsilon, \alpha))| \\ &\le |\mathcal{P}_n(c+\epsilon, \alpha)| + 2|\mathcal{P}_{n+1}(c+\epsilon, \alpha)| + \cdots + (N+1)|\mathcal{P}_{n+N}(c+\epsilon, \alpha)|.\end{aligned}$$

For all n then there is a number $k(n) \in \{0, 1, \cdots, N\}$ such that

$$\frac{|\mathcal{B}_n(c, \alpha)|}{N+1} \le \big(k(n)+1\big)|\mathcal{P}_{n+k(n)}(c+\epsilon, \alpha)|.$$

Thus

$$\frac{1}{n+k(n)} \log |\mathcal{P}_{n+k(n)}(c+\epsilon, \alpha)| \ge \frac{1}{n+k(n)} \log \frac{|\mathcal{B}_n(c, \alpha)|}{(N+1)(k(n)+1)} \tag{4.7}$$

and the right-hand side of (4.7) converges to $h(X(c, \alpha))$ as $n \to \infty$ since $\mathcal{B}_n(X(c, \alpha)) = \mathcal{B}_n(c, \alpha)$ and $k(n) \le N$ and N is finite. Therefore

$$\limsup_{n \to \infty} \frac{1}{m} \log |\mathcal{P}_m(c+\epsilon, \alpha)| \ge h(X(c, \alpha)) \qquad (\epsilon > 0),$$

so that

$$\limsup_{n \to \infty} \frac{1}{m} \log |\mathcal{P}_m(c, \alpha)| \ge \lim_{\epsilon \to 0^+} h(X(c-\epsilon, \alpha)) = h(X(c, \alpha))$$

and $X(c, \alpha)$ is periodic saturated. □

Finally we prove Theorem 4.1.5.

Proof of Theorem 4.1.5. Let $c > \max\{2/ + |1+\alpha|, \sqrt{2}+3\}$, and let α be transcendental over $\mathbb{Z}$. To obtain a contradiction, suppose that $X(c,\alpha)$ contains a shift Y of finite type with positive entropy. We may assume that Y is irreducible, and that $Y = \mathbf{X}_G$ by considering the higher block system if necessary. Since $h(Y) > 0$, we can assume that there is a vertex I such that two edges start at I. There are two different paths u, v on G such that u and v start and end at I, $|u| = |v| = l$. Since α is transcendental over $\mathbb{Z}$, we obtain $s(u) \neq s(v)$.

We will find a finite concatenation U of u's and v's such that $s(U) > c$. Since u and v start and end at I, the point U^∞ must be in Y, and in $X(c,\alpha)$. However, since $s(U) > c$, U^∞ can not be in $X(c,\alpha)$, it is a contradiction. Now we construct U. Since $u^\infty, v^\infty \in Y \subseteq X(c,\alpha)$, we have $|s(u)|, |s(v)| < c$. Also, since $s(u) \neq s(v)$, $0 < |s(v) - s(u)| < 2c$. We choose a positive number k with $k > 3c/|s(v) - s(u)|$. Let

$$U = u^{m_1} v u^{m_2} v \cdots u^{m_k} v \qquad (m_1, \cdots, m_k \in \mathbb{N}).$$

Note that, for any blocks w_1, w_2, w_3, w_4 with $|w_2| = |w_4|$,

$$\begin{aligned} s(w_1 w_2 w_3) &= s(w_1) + \alpha^{|w_1|} s(w_2) + \alpha^{|w_1 w_2|} s(w_3) \\ &= s(w_1) + \alpha^{|w_1|} s(w_2) + \alpha^{|w_1 w_2|} s(w_3) + \alpha^{|w_1|} s(w_4) - \alpha^{|w_1|} s(w_4) \\ &= s(w_1 w_4 w_3) + \alpha^{|w_1|}(s(w_2) - s(w_4)), \end{aligned} \tag{4.8}$$

and that $|s(u^{m_1+\cdots+m_k+k})| \leq r(u^\infty) < c$. Applying k times (4.8) to $s(U)$ we obtain

$$\begin{aligned} s(U) &= s(u^{m_1} u u^{m_2} v \cdots u^{m_k} v) + \alpha^{l_1}(s(v) - s(u)) \\ &= \cdots \\ &= s(u^{m_1} u u^{m_2} u \cdots u^{m_k} v) + \alpha^{l_1}(s(v) - s(u)) + \alpha^{l_1 + 1_2 + l}(s(v) - s(u)) \\ &\qquad + \cdots + \alpha^{l_1 + \cdots + l_k + (k-1)l}(s(v) - s(u)) \\ &= s(u^{m_1 + \cdots + m_k + k}) \\ &\qquad + \big(s(v) - s(u)\big)\big(\alpha^{l_1} + \alpha^{l_1 + l_2 + l} + \cdots + \alpha^{l_1 + \cdots + l_k + (k-1)l}\big), \end{aligned} \tag{4.9}$$

where $l_j = m_j l\,(j = 1, \cdots, k)$. Let $S(\alpha)$ be the last term of (4.9), i.e.,

$$S(\alpha) = \alpha^{l_1} + \alpha^{l_1+l_2+l} + \cdots + \alpha^{l_1+\cdots+l_k+(k-1)l}.$$

Thus

$$\begin{aligned} |s(U)| &\geq \big||s(v) - s(u)||S(\alpha)| - | - s(u^{m_1+\cdots+m_k+k})|\big| \\ &\geq |s(v) - s(u)||S(\alpha)| - |s(u^{m_1+\cdots+m_k+k})| \\ &> |s(v) - s(u)||S(\alpha)| - c. \end{aligned} \tag{4.10}$$

Let $0 < \epsilon < 1/2$. We choose m_1 so that $|\alpha^{l_1} - 1| < \epsilon/k$, and choose m_2 with $|\alpha^{l_1+l_2+l} - 1| < \epsilon/k$. Continuing this process, there are $m_1, m_2, \cdots, m_k$ such that

$$\left|\alpha^{l_1+l_2+\cdots+l_j+(j-1)l} - 1\right| < \epsilon/k \qquad (j = 1, \cdots, k),$$

then $|S(\alpha) - k| < \epsilon$. From the equation (4.10),

$$\begin{aligned} |s(U)| &> (k - \epsilon)|s(v) - s(u)| - c \\ &> 2c - \epsilon|s(v) - s(u)| \end{aligned}$$

so that $|s(U)| > c$ since $\epsilon < 1/2$ and $|s(v) - s(u)| < 2c$. The proof is done. $\square$

4.2 Flips for the disk system $X(c, \alpha)$

In this section we show that $X(c, \alpha)$ given in Theorem 4.1.3 satisfies the remaining properties (1), (2) and (7) of Theorem 4.0.1. The following theorem shows that $X(c, \alpha)$ satisfies Theorem 4.0.1 (1) and (2).

Theorem 4.2.1. *If $c > 2/|1 + \alpha|$, then $X(c, \alpha)$ is irreducible and periodic points dense. In addition, if $c > \max\{2/|1 - \alpha|,\ 2/|1 + \alpha|\}$, then $X(c, \alpha)$ is infinite.*

Proof. Suppose that $c > 2/|1+\alpha|$. Let $x \in X(c, \alpha)$ and $n > 0$. There is a block $u \in \mathcal{B}(c, \alpha)$ such that (i) $x_{[-n,n]}$ occurs in u; (ii) $u^\infty = uu \cdots \in W(c, \alpha)$ (Lemma 4.1.7). We can write $u = u_0 \cdots u_k$ and $u_i \cdots u_{i+2n} = x_{[-n,n]}$ for $0 \leq i \leq k - 2n$.

Let $y = \cdots uuu \cdots$ with $y_{[-i-n,k-i-n+1]} = u$. Then y is a periodic point in $X(c,\alpha)$ and $x_{[-n,n]} = y_{[-n,n]}$, so that $X(c,\alpha)$ is periodic points dense.

To prove that the irreducibility of $X(c,\alpha)$ it is enough to show that whenever $x, y \in W(c,\alpha)$ and $n, m \in \mathbb{N}$, then $x_{[0,n-1]}y_{[0,m-1]} \in \mathcal{B}(c,\alpha)$. The proof is similar to the one of Lemma 4.1.7. Suppose that $x, y \in W(c,\alpha)$ and $n, m \in \mathbb{N}$. Let

$$\delta = \frac{1}{9}\big(c - \max\{2/|1+\alpha|, r(x), r(y)\}\big),$$

then $\delta > 0$. We may assume that $|s_n(x)|, |s_m(y)| < \delta$ since x, y are recurrent. If $|1 - \alpha^{n+m}| \geq \sqrt{2}$, let $z = (x_{[0,n-1]}y_{[0,m-1]})^\infty \in \{-1,1\}^{\mathbb{N}}$. It suffices to check that $r(z) < c$. Since

$$s_k(z) = \begin{cases} s_k(x) & \text{if } 0 \leq k \leq n \\ s_n(x) + \alpha^n s_{k-n}(y) & \text{if } n+1 \leq k \leq n+m, \end{cases}$$

we have for $0 \leq r \leq n+m-1$

$$|s_r(z)| \leq \delta + \max\{r(x), r(y)\} \quad \text{and} \quad |s_{n+m}(z)| \leq 2\delta.$$

From Lemma 4.1.1

$$\begin{aligned} |s_{(n+m)q+r}(z)| &\leq 2\sqrt{2}\delta + \delta + \max\{r(x), r(y)\} \\ &< 5\delta + \max\{r(x), r(y)\}, \end{aligned}$$

where $q \in \mathbb{N}$ and $0 \leq r \leq n+m-1$. Thus $r(z) < 9\delta + \max\{r(x), r(y)\} \leq c$, so $z \in W(c,\alpha)$.

Suppose that $|1 - \alpha^{n+m}| < \sqrt{2}$. By Lemma 4.1.2 there is an odd number $N > 0$ such that $|1 - \alpha^{n+m+N}| > 1$ and $2/|1+\alpha| < \delta$. Let $l = n+m$. We define $z' \in \{-1,1\}^{\mathbb{N}}$ as follows: let $z' = (z'_{[0,l+N-1]})^\infty$ with $z'_{[0,l+N-1]} = x_{[0,n-1]}y_{[0,m-1]}1(-1)\cdots 1(-1)1$. It suffices to check that $r(z') < c$. As in the proof of Lemma 4.1.7, it is easily seen that

$$s_k(z') = \begin{cases} s_k(z) & \text{if } 0 \leq k \leq l \\ s_l(z) + [\alpha^l + (-1)^{k-l-1}\alpha^k]/(1+\alpha) & \text{if } l+1 \leq k \leq l+N. \end{cases}$$

Thus for $0 \leq r \leq l+N-1$,

$$|s_r(z')| \leq 2\delta + \max\{2/|1+\alpha|, r(x), r(y)\} \quad \text{and} \quad |s_{l+N}(z')| \leq 3\delta.$$

From Lemma 4.1.1

$$\begin{aligned}|s_{(l+N)q+r}(z')| &\leq 6\delta + 2\delta + \max\{2/|1+\alpha|, r(x), r(y)\} \\ &= 8\delta + \max\{2/|1+\alpha|, r(x), r(y)\}.\end{aligned}$$

where $q \in \mathbb{N}$ and $0 \leq r \leq n+m-1$. By assumption of δ, $r(z') < c$ and $z' \in W(c,\alpha)$.

Finally we show that $X(c,\alpha)$ is infinite. Let $c > \max\{2/|1+\alpha|, 2/|1-\alpha|\}$. For all $m, n \in \mathbb{N}$, $[1(-1)]^m$ and 1^n are in $\mathcal{B}(c,\alpha)$. Indeed, let $x = [1(-1)]^\infty$ and $y = 1^\infty$. Since x, y are periodic, both x, y are recurrent. Also $r(x) = 2/|1+\alpha| < c$ and $r(y) = 2/|1-\alpha| < c$, therefore $x, y \in W(c,\alpha)$. For the moment let $a = 1$ and $b = -1$. Let $w_1 = ab \in \mathcal{B}(c,\alpha)$. By the irreducibility of $X(c,\alpha)$ there are $u_1, u_2, \cdots$ such that

$$w_n = a^{|w_{n-1}|} u_{n-1} ab \in \mathcal{B}(c,\alpha) \quad (n = 2, 3, \cdots).$$

For each n, there is a point $x^{(n)} \in X(c,\alpha)$ with $x^{(n)}_{[0,|w_n|-1]} = w_n$. Then $x^{(n)} \neq x^{(m)}$ if $n \neq m$. Indeed, let $n < m$. By definitions $x^{(m)}_{[0,|w_n|-1]} = a^{|w_{n-1}|}$ ends with a, but $x^{(n)}_{[0,|w_n|-1]} = w_n$ ends with b. Thus $X(c,\alpha)$ has infinitely many points. In fact, $X(c,\alpha)$ is a perfect set (i.e., $X(c,\alpha)$ is closed and every point of $X(c,\alpha)$ is a limit point of $X(c,\alpha)$) so that it is uncountable. □

Finally we prove that $X(c,\alpha)$ given in Theorem 4.1.3 has a flip (Theorem 4.2.2). In particular we show that $X(c,\alpha)$ is closed under the mirror map, and that there is an automorphism of order 2 of $X(c,\alpha)$.

We need some notations. For $w = w_0 w_1 \cdots w_{k-2} w_{k-1} \in \{-1, 1\}^k$, let

$$\begin{aligned}\widetilde{w} &= w_{k-1} w_{k-2} \cdots w_1 w_0, \\ -w &= (-w_0)(-w_1) \cdots (-w_{k-2})(-w_{k-1}).\end{aligned}$$

Then

$$\begin{aligned} s(\widetilde{w}) &= \alpha^{k-1}\bar{\alpha}^{k-1}(w_{k-1} + \alpha w_{k-2} \cdots \alpha^{k-1} w_0) \\ &= \alpha^{k-1}(w_0 + \bar{\alpha} w_1 \cdots \bar{\alpha}^{k-1} w_{k-1}) = \alpha^{k-1}\overline{s(w)} \end{aligned}$$

and

$$s(-w) = (-w_0) + \alpha(-w_1) \cdots + \alpha^{k-1}(-w_{k-1}) = -s(w),$$

so that

$$|s(\widetilde{w})| = |s(w)| \qquad \text{and} \qquad |s(-w)| = |s(w)|. \tag{4.11}$$

We define a map $\theta : \{-1,1\}^{\mathbb{Z}} \to \{-1,1\}^{\mathbb{Z}}$ by

$$\theta(x)_i = -x_i \qquad (x \in \{-1,1\}^{\mathbb{Z}},\ i \in \mathbb{Z}).$$

It is trivial that the mirror map ρ is a flip for $\{-1,1\}^{\mathbb{Z}}$, and that θ is an automorphism of $\{-1,1\}^{\mathbb{Z}}$ with $\theta^2 = \mathrm{id}$, and that $\rho\theta = \theta\rho$. Thus $\rho\theta$ also is a flip for $\{-1,1\}^{\mathbb{Z}}$.

Theorem 4.2.2. *If $c > 2/|1+\alpha|$, then $X(c,\alpha)$ is closed under both the mirror map ρ and the map θ. Thus both ρ and $\rho\theta$ are flips for $X(c,\alpha)$.*

Proof. Let $c > 2/|1+\alpha|$. To prove $\theta(X(c,\alpha)) \subseteq X(c,\alpha)$ let $x \in X(c,\alpha)$ and $y = \theta(x)$. It is clear that $y \in X(c,\alpha)$ since the second relation of (4.11) implies that $|s_k(y)| = |s_k(x)|$, $k \in \mathbb{N}$.

To prove $X(c,\alpha)$ is closed under the mirror map ρ it is enough to show that whenever $w \in \mathcal{B}(c,\alpha)$ then $\widetilde{w} \in \mathcal{B}(c,\alpha)$.

Suppose that $w \in \mathcal{B}(c,\alpha)$. From Lemma 4.1.1 there are $x \in X(c,\alpha)$ and $n > 0$ such that $x = (x_{[0,n-1]})^\infty$ and w occurs in $x_{[0,n-1]}$. Let

$$\delta = \frac{1}{2}(c - \max\{2/|1+\alpha|,\ r(x)\}) > 0.$$

Let $y = (\widetilde{x_{[0,n-1]}})^\infty \in \{-1,1\}^{\mathbb{N}}$. Then $\widetilde{w}$ occurs in y and y is recurrent. It remains to show that $r(y) < c$.

Let $k \in \mathbb{N}$. As in the proof of Lemma 4.1.1, there is a positive number m

such that $mn > k$ and $|s_{mn}(y)| < \delta$. We can write

$$y_{[0,mn-1]} = y_{[0,k-1]}(x_r \cdots x_1 x_0)(x_{n-1} \cdots x_0)^d$$

where $k + r + dn + 1 = mn$, $0 \leq r \leq n-1$ and $d \geq 0$.

Since $y_{[k,mn-1]} = \widetilde{x_{[0,dn+r]}}$, the first relation of (4.11) implies that $|s(y_{[k,mn-1]})| \leq r(x)$. Since

$$\begin{aligned} s_{mn}(y) &= s_k(y) + \alpha^k(y_k + \alpha y_{k+1} + \cdots + \alpha^{mn-k-1} y_{mn-1}) \\ &= s_k(y) + \alpha^k s(y_{[k,mn-1]}), \end{aligned}$$

we obtain

$$|s_k(y)| \leq |s_{mn}(y)| + |s(y_{[k,mn-1]})| < \delta + r(x).$$

Then $r(y) \leq \delta + r(x) < c$ and $y \in X(c, \alpha)$, so that $\widetilde{w} \in \mathcal{B}(c, \alpha)$. $\square$

Suppose that $c > 0$, and that $\alpha \in \mathbb{C}$, $|\alpha| = 1$, $\alpha^n \neq 1$ for all $n \geq 1$. If $c > \max\{2/|1+\alpha|,\ 2/|1-\alpha|,\ \sqrt{2}+3\}$ and α is transcendental over $\mathbb{Z}$, then the disk system $X(c, \alpha)$ given in Theorem 4.1.3 satisfies the properties of Theorem 4.0.1 by Theorem 4.1.3, 4.1.4, 4.2.1, 4.2.2 and Corollary 4.1.6.

Chapter 5

Some results of S-gap shifts and $\mathbf{X}(\xi, \xi', X)$

In this chapter we compute the zeta function and the entropy of S-gap shifts (Section 5.1). We use the result to construct a synchronized system which is not periodic saturated (Section 5.2). For the construction, we consider an S-gap shift X and an irreducible shift Y, and change an allowed 10^s1 in X with $1y_{[0,s-1]}1$ in $\mathcal{B}(Y)$ where y contains all blocks in $\mathcal{B}(Y)$. The symbol 1 is a finitary block for (Y, σ_Y). In the last section we generalize the construction. We find two right-infinite sequences ξ, ξ' and an irreducible shift space X over $\{0, 1\}$, and then change 0^m and 1^n appearing in a point of a dense G_δ subset of X with $\xi_{[0,m-1]}$ and $\xi'_{[0,n-1]}$, respectively. This method induces a new shift space $\mathbf{X}(\xi, \xi', X)$ and we will survey properties of $\mathbf{X}(\xi, \xi', X)$ in Section 5.3.

5.1 The entropy and the zeta function of S-gap shifts

Let S be a subset of $\{0, 1, 2, 3, \cdots\}$. We recall that the S-gap shift $\mathbf{X}(S)$ is the closure of the set of all bi-infinite concatenations of blocks from $\{10^s : s \in S\}$. We will compute the entropy and the zeta function of $\mathbf{X}(S)$. We will find the

unique solution to
$$1 = \sum_{s \in S} t^{s+1}$$
instead of directly computing $|\mathcal{B}_n(\mathbf{X}(S))|$ ([LinM, Exercise 4.3.7], [Spa]). If S is finite, then $\sum_{s\in S} t^{s+1}$ is a polynomial; otherwise it has the radius 1 of convergence. Then there is the unique number $\lambda \in (0,1)$ with $1 = \sum_{s\in S} \lambda^{s+1}$. If $|t| < \lambda$, then $|\sum_{s\in S} t^{s+1}| < 1$, and λ is a simple pole of $1/(1 - \sum_{s\in S} t^{s+1})$. We set

$$\begin{aligned}
&A = \{10^s : s \in S\}, \quad \mathcal{L} = \bigcup_{n=0}^{\infty} A^n, \\
&\mathcal{L}_n = \mathcal{L} \cap \mathcal{B}_n(\mathbf{X}(S)) \quad (n = 0, 1, 2, \cdots) \ \ \textit{and} \\
&\mathcal{L}_n^k = \mathcal{L}_n \cap A^k \quad (n, k = 0, 1, 2, \cdots).
\end{aligned}$$

It is clear that $\mathcal{L} \subseteq \mathcal{B}(\mathbf{X}(S))$ and $\{\mathcal{L}_n : n = 0, 1, 2, \cdots\}$ is a partition of $\mathcal{L}$. If $w \in A^k$, then w contains exactly k 1's. Hence $\mathcal{L}_n^k = \emptyset$ if $k > n$. Also $\{\mathcal{L}_n^k : 0 \leq k \leq n\}$ is a partition of $\mathcal{L}_n$.

Proposition 5.1.1. *If $|t| < \lambda$, then*
$$\sum_{n=0}^{\infty} |\mathcal{L}_n| t^n = \frac{1}{1 - \sum_{s\in S} t^{s+1}}.$$

Proof. Let $|t| < \lambda$. We use the following lemma which is proved in the last paragraph of this proof.

Lemma 5.1.2. *Let $k = 0, 1, 2, \cdots$. Then*
$$\left(\sum_{s\in S} t^{s+1} \right)^k = \sum_{n=k}^{\infty} |\mathcal{L}_n^k| t^n. \tag{5.1}$$

Suppose that $0 \le t < \lambda$. Then by Lemma 5.1.2

$$\begin{aligned}\frac{1}{1-\sum_{s\in S} t^{s+1}} &= \sum_{k=0}^{\infty}\Big(\sum_{s\in S} t^{s+1}\Big)^k = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} |\mathcal{L}_n^k| t^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} |\mathcal{L}_n^k| t^n = \sum_{n=0}^{\infty} |\mathcal{L}_n| t^n.\end{aligned}$$

From the identity theorem for holomorphic functions, we obtain the desired result when $|t| < \lambda$.

We now finish this proof by proving Lemma 5.1.2. We proceed by induction on k. Let $k = 0$. Since $\mathcal{L}_0^0 = \{\epsilon\}$ (ϵ is the empty block) and $\mathcal{L}_n^0 = \emptyset$ for $n \ge 1$, the equality (5.1) hold. Suppose that (5.1) is true for a k. Since $|t| < \lambda$ and $|\mathcal{L}_j^k| = 0$ for $k > j$, it follows that

$$\begin{aligned}\left(\sum_{s\in S} t^{s+1}\right)^{k+1} &= \sum_{n=k}^{\infty} |\mathcal{L}_n^k| t^n \sum_{n=1}^{\infty} |\mathcal{L}_n^1| t^n = \sum_{n=1}^{\infty}\sum_{m=1}^{n} |\mathcal{L}_{n+1-m}^k||\mathcal{L}_m^1| t^{n+1} \\ &= \sum_{n=k}^{\infty}\sum_{m=1}^{n-k+1} |\mathcal{L}_{n+1-m}^k||\mathcal{L}_m^1| t^{n+1} = \sum_{n=k+1}^{\infty}\sum_{m=1}^{n-k} |\mathcal{L}_{n-m}^k||\mathcal{L}_m^1| t^n \\ &= \sum_{n=k+1}^{\infty} |\mathcal{L}_n^{k+1}| t^n,\end{aligned}$$

where the last step comes from the following: let $n \ge k+1$. The map

$$\bigcup_{m=1}^{n-k} \mathcal{L}_m^1 \times \mathcal{L}_{n-m}^k \ni (10^{m-1}, 10^{s_1}\cdots 10^{s_k}) \mapsto 10^{m-1}10^{s_1}\cdots 10^{s_k} \in \mathcal{L}_n^{k+1}$$

is a one-to-one correspondence; hence

$$|\mathcal{L}_n^{k+1}| = \sum_{m=1}^{n-k} |\mathcal{L}_m^1||\mathcal{L}_{n-m}^k| \quad (n \ge k+1).$$

This completes the proof of the lemma. $\square$

Proposition 5.1.1 provides $\limsup_{n\to\infty} |\mathcal{L}_n|^{1/n} = 1/\lambda$. Since $\mathcal{L}_n \subseteq \mathcal{B}_n(\mathbf{X}(S))$, we have $\lim_{n\to\infty} |\mathcal{B}_n(\mathbf{X}(S))|^{1/n} \geq 1/\lambda$. We claim that $\lim_{n\to\infty} |\mathcal{B}_n(\mathbf{X}(S))|^{1/n} = 1/\lambda$.

Theorem 5.1.3. *The entropy of* $\mathbf{X}(S)$ *is* $\log(1/\lambda)$.

Proof. Let $\delta > 0$. We will show that

$$\frac{1}{\lambda} \leq \lim_{n\to\infty} |\mathcal{B}_n(\mathbf{X}(S))|^{1/n} \leq \frac{1}{\lambda} + \delta.$$

Then $\lim_{n\to\infty} |\mathcal{B}_n(\mathbf{X}(S))|^{1/n} = 1/\lambda$ and $h(\mathbf{X}(S)) = \log(1/\lambda)$. The first inequality is proved as the above argument.

To prove the second inequality we suppose that $\mu = \delta + (1/\lambda)$. There is a constant $c > 1$ such that $|\mathcal{L}_n| \leq c\mu^n$ for all n. For each n

$$\mathcal{B}_n(\mathbf{X}(S)) = \{0^n\} \cup (\mathcal{B}_n(\mathbf{X}(S)) \setminus \{0^n\}).$$

If $w \in \mathcal{B}_n(\mathbf{X}(S)) \setminus \{0^n\}$, then $w = 0^m u 1 0^k$ where $u \in \bigcup_{j=0}^{n-1} \mathcal{L}_j$, $0 \leq m, k \leq n-1$ and $m + k + 1 + |u| = n$. For each $u \in \mathcal{L}_j$ there are $(n-j)$ blocks, so that $|\mathcal{B}_n(\mathbf{X}(S)) \setminus \{0^n\}| \leq \sum_{j=0}^{n-1} (n-j)|\mathcal{L}_j|$. Then

$$\begin{aligned}
|\mathcal{B}_n(\mathbf{X}(S))| &\leq 1 + \sum_{j=0}^{n-1}(n-j)|\mathcal{L}_j| \leq 1 + \sum_{j=0}^{n-1}(n-j)c\mu^j \\
&= 1 + cn + c\mu(n-1) + \cdots + c\mu^{n-1} \\
&= c\mu^n \left(\frac{1}{c\mu^n} + \frac{n}{\mu^n} + \frac{n-1}{\mu^{n-1}} + \cdots + \frac{1}{\mu} \right) \\
&< c\mu^n (1 + n + (n-1) + \cdots + 2 + 1) \\
&\leq c\mu^n \, \frac{2n + n(n+1)}{2} \\
&= \frac{c}{2} n(n+3)\mu^n
\end{aligned}$$

where the fourth step comes from the assumptions $c > 1$ and $\mu > 1/\lambda > 1$. Therefore $\lim_{n\to\infty} |\mathcal{B}_n(\mathbf{X}(S))|^{1/n} \leq \mu = \delta + (1/\lambda)$, and the proof is done. $\square$

Recall that the zeta function of a shift space X is given by

$$\zeta_X(t) = \exp\left(\sum_{n=1}^{\infty} \frac{p_n(X)}{n} t^n\right) \tag{5.2}$$

where $p_n(X) = |\{x : \sigma_X^n(x) = x\}|$. In the rest of this section, we compute this function when $X = \mathbf{X}(S)$ for some S.

Theorem 5.1.4. *Let S be a subset of $\{0, 1, 2, 3, \cdots\}$. Then*

$$\zeta_{\mathbf{X}(S)}(t) = \begin{cases} \dfrac{1}{1 - \sum_{s \in S} t^{s+1}} & \text{if } |S| < \infty \\ \dfrac{1}{(1 - \sum_{s \in S} t^{s+1})(1 - t)} & \text{if } |S| = \infty. \end{cases}$$

It is known that $0^\infty \in \mathbf{X}(S)$ if and only if $|S| = \infty$. In the rest of this section we denote $\sigma_{\mathbf{X}(S)}$ briefly by σ_S. If we put

$$q_n = |\{x \in \mathbf{X}(S) : \sigma_S^n(x) = x, \ x \neq 0^\infty\}| \qquad (n = 1, 2, 3, \cdots),$$

then $p_n(\mathbf{X}(S)) = q_n$ in the case when S is finite and $p_n(\mathbf{X}(S)) = q_n + 1$ in the case when S is infinite. By definition, if S is finite,

$$\zeta_{\mathbf{X}(S)}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{q_n}{n} t^n\right)$$

and if S is infinite,

$$\zeta_{\mathbf{X}(S)}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{q_n + 1}{n} t^n\right) = \frac{1}{1-t} \exp\left(\sum_{n=1}^{\infty} \frac{q_n}{n} t^n\right).$$

We shall have established Theorem 5.1.4 if we prove the following:

$$\exp\left(\sum_{n=1}^{\infty} \frac{q_n}{n} t^n\right) = \frac{1}{1 - \sum_{s \in S} t^{s+1}}. \tag{5.3}$$

Let $\gamma : G \times M \to M$ be a group action of G on M and $x \in M$. The orbit $Orb_\gamma(x)$ of x and the isotropy group $I_\gamma(x)$ of x are given by

$$Orb_\gamma(x) = \{\gamma(g, x) : g \in G\} \quad \text{and} \quad I_\gamma(x) = \{g \in G : \gamma(g, x) = x\},$$

respectively. Let $1 \le m \le n$. If we put

$$P_n^m = \{x \in \mathbf{X}(S) : \sigma_S^n(x) = x \text{ and } |\{i \in [0, n) : x_i = 1\}| = m\},$$

then P_n^m, $1 \le m \le n$, are mutually disjoint, and $q_n = \sum_{m=1}^n |P_n^m|$.

Lemma 5.1.5. *For $1 \le m \le n$, $n|\mathcal{L}_n^m| = m|P_n^m|$.*

Proof. Let $1 \le m \le n$. We define

$$\begin{aligned}
&\gamma : \mathbb{Z}/n\mathbb{Z} \times P_n^m \ni (j + n\mathbb{Z}, x) \mapsto \sigma_S^j(x) \in P_n^m \\
&\pi : \mathcal{L}_n^m \ni 10^{s_1}10^{s_2} \cdots 10^{s_m} \mapsto 10^{s_2} \cdots 10^{s_m}10^{s_1} \in \mathcal{L}_n^m \\
&\tau : \mathbb{Z}/m\mathbb{Z} \times \mathcal{L}_n^m \ni (j + m\mathbb{Z}, w) \mapsto \pi^j(w) \in \mathcal{L}_n^m,
\end{aligned}$$

then γ, π, τ are well-defined and $\pi^m = \mathrm{id}_{\mathcal{L}_n^m}$. If we put

$$\mathcal{P} = \{Orb_\gamma(x) : x \in P_n^m\} \quad \text{and} \quad \mathcal{Q} = \{Orb_\tau(w) : w \in \mathcal{L}_n^m\},$$

it is clear that each $\mathcal{P}$ and $\mathcal{Q}$ is a partition of P_n^m and $\mathcal{L}_n^m$, respectively. We claim that there is a one-to-one correspondence between $\mathcal{Q}$ and $\mathcal{P}$. For a block u, let u^∞ denote the right-infinite sequence $\cdots u.uu \cdots$. We define a map Γ from $\mathcal{Q}$ into $\mathcal{P}$ by $\Gamma(Orb_\tau(w)) = Orb_\gamma(w^\infty)$. If $w, v \in \mathcal{L}_n^m$, then it is obvious that $w^\infty \in P_n^m$, and that $Orb_\tau(w) = Orb_\tau(v)$ if and only if $Orb_\gamma(w^\infty) = Orb_\gamma(v^\infty)$; hence Γ is well-defined and one-to-one. If $x \in P_n^m$, there are $j \in [0, n)$ and $w \in \mathcal{L}$ such that $\sigma_S^j(x) = w^\infty$. Then $Orb_\gamma(x) = Orb_\gamma(w^\infty)$ and Γ is onto. It thus follows that there are $w^{(1)}, \cdots, w^{(k)} \in \mathcal{L}_n^m$ such that

$$P_n^m = \bigcup_{i=1}^k Orb_\gamma((w^{(i)})^\infty) \;\text{ and }\; \mathcal{L}_n^m = \bigcup_{i=1}^k Orb_\tau(w^{(i)}) \tag{5.4}$$

It is well-known that, for a group action γ' of G on M and $x \in M$, $|Orb_{\gamma'}(x)| = |G/I_{\gamma'}(x)|$. Since $|I_\gamma(w^\infty)| = |I_\tau(w)|$ for $w \in \mathcal{L}_n^m$ (by definitions of π and u^∞), we have

$$\frac{1}{n}|Orb_\gamma(w^\infty)| = \frac{1}{m}|Orb_\tau(w)| \qquad (w \in \mathcal{L}_n^m).$$

Combining this equality with (5.4) we obtain the desired result. $\square$

We conclude this section by proving (5.3). It suffices to show that (5.3) is true for $0 < t < \lambda$ by the identity theorem for holomorphic functions. Let $0 < t < \lambda$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{q_n}{n} t^n &= \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{|P_n^m|}{n} t^n = \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \frac{|P_n^m|}{n} t^n \\
&= \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \frac{|\mathcal{L}_n^m|}{m} t^n = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=m}^{\infty} |\mathcal{L}_n^m| t^n \\
&= \sum_{m=1}^{\infty} \frac{1}{m} \left(\sum_{s \in S} t^{s+1} \right)^m = -\log \left(1 - \sum_{s \in S} t^{s+1} \right)
\end{aligned}$$

where the third and fifth step come from Lemma 5.1.5 and Lemma 5.1.2, respectively. Hence we obtain (5.3), and the proof of Theorem 5.1.4 is complete.

5.2 A synchronized system which is not periodic saturated

We consider an S-gap shift with positive entropy, and construct an irreducible shift space whose entropy is positive but it has no periodic points. We then combine these two shift spaces and obtain the following:

Theorem 5.2.1. *There is a synchronized system which is not periodic saturated.*

From now on let $\mathbb{N}_0$ denote the set $\mathbb{N}\setminus\{0\}$. Let $1/2 \le \delta < 1$. The map given by $\{0,1\}^{\mathbb{N}_0} \ni x \mapsto \sum_{m=0}^{\infty} x_m \delta^m \in [0,\ 1/(1-\delta)]$ is continuous. Also it is onto from the following lemma.

Lemma 5.2.2. *Let $1/2 \le \delta < 1$ and $I = (\delta/(1-\delta), 1/(1-\delta)]$. We define a map Ψ on $[0,\ 1/(1-\delta)]$ by $\Psi(s) = \delta^{-1}(s - \chi_I(s))$ where χ_I is the characteristic function on I. Then*

(1) $\Psi([0,\ 1/(1-\delta)]) = [0,\ 1/(1-\delta)]$ *and*

(2) $s = \sum_{m=0}^{\infty} \delta^m \chi_I(\Psi^n(s))$ *for all* $s \in [0,\ 1/(1-\delta)]$.

Proof. By the definition of Ψ, (1) hold. Observe that if $s \in [0,\ 1/(1-\delta)]$ then $s = \chi_I(s) + \delta\Psi(s)$. Combining this observation with (1) gives

$$\Psi^n(s) = \chi_I(\Psi^n(s)) + \delta\Psi^{n+1}(s)$$

where $s \in [0,\ 1/(1-\delta)]$ and $n = 0, 1, 2, \cdots$. Then we have

$$s = \sum_{n=0}^{N} \delta^n \chi_I(\Psi^n(s)) + \delta^{N+1}\Psi^{N+1}(s)$$

for all N by induction on N. Since $\delta^{N+1}\Psi^{N+1}(s) \to 0$ as $N \to \infty$ we obtain the desired result. $\square$

Let $1/2 \le \delta < 1$. Since $[0,1] \subseteq [0,\ \delta/(1-\delta)]$, Lemma 5.2.2 implies that there is a right-infinite sequence $x \in \{0,1\}^{\mathbb{N}_0}$ such that $\sum_{m=1}^{\infty} x_m \delta^m = 1$. We set

$$S = \{m \in \mathbb{N} : x_{m+1} = 1\}.$$

If x has finitely many the symbol 1, there is a positive number M such that $x_M = 1$ and $x_m = 0$ for $m \ge M+1$. Then

$$1 = \sum_{m=1}^{\infty} x_m \delta^m = \sum_{m=1}^{M} x_m \delta^m. \tag{5.5}$$

Repeated application of 5.5 shows that $\sum_{m=1}^{\infty} x_m\delta^m = \sum_{m=1}^{M} x_m\delta^m$ can be presented by $\sum_{n=1}^{\infty} y_n\delta^n$ where $y_n = 1$ for infinitely many n:

$$\begin{aligned}1 = \sum_{m=1}^{M} x_m\delta^m &= \sum_{m=1}^{M-1} x_m\delta^m + x_M\delta^M\left(\sum_{m=1}^{M} x_m\delta^m\right)\\ &= \sum_{m=1}^{M-1} x_m\delta^m + x_M\sum_{m=1}^{M-1} x_m\delta^{m+M} + x_M^2\delta^{2M}\left(\sum_{m=1}^{M} x_m\delta^m\right)\\ &= \cdots .\end{aligned}$$

Hence we may assume that S is infinite. Thus

$$\sum_{s\in S}\delta^{s+1} = \sum_{n=0}^{\infty}\chi_S(n)\delta^{n+1} = \sum_{m=1}^{\infty} x_m\delta^m = 1$$

and there is the S-gap shift with entropy $\log(1/\delta)$. We have thus proved the following:

Theorem 5.2.3. *If $\delta\in[1/2,\ 1)$, there is an infinite subset S of $\mathbb{N}$ such that the S-gap shift $\mathbf{X}(S)$ has entropy $\log(1/\delta)$.*

Let $\alpha\in(0,1)\setminus\mathbb{Q}$ and $\mathcal{A}=\{-1,\alpha\}$. In Section 4.1 we use the notation α as a constraint of the construction of a disk system. In this section α is a symbol in the alphabet $\mathcal{A}$. For $x\in\mathcal{A}^{\mathbb{N}_0}$ and $n=1,2,3\cdots$, let

$$\Sigma_n(x) = x_1 + x_2 + \cdots + x_n \quad r(x) = \sup\{|\Sigma_n(x)| : n = 1,2,\cdots\}.$$

For a number β we put $\lfloor\beta\rfloor = \max\{m\in\mathbb{Z} : m\le\beta\}$. We choose inductively positive integers $n_1, n_2, n_3, \cdots$ such that

$$\lfloor n_1\alpha\rfloor = 1 \quad\text{and}\quad \left\lfloor\sum_{j=1}^{k-1}(n_j\alpha-1)+n_k\alpha\right\rfloor = 1\ (k\ge 2).$$

We define a right-infinite sequence $\mu \in \mathcal{A}^{\mathbb{N}_0}$ by

$$\mu = \alpha^{n_1}(-1)\alpha^{n_2}(-1)\alpha^{n_3}(-1)\cdots .$$

Then the closure of $\{\Sigma_n(\mu) : n \in \mathbb{N}_0\}$ is equal to $[0, 1+\alpha]$ and $r(\mu) = 1+\alpha$. Let $c > 2$. We set

$$W(\alpha, c) = \{x \in \mathcal{A}^{\mathbb{N}_0} : r(x) < c\} \text{ and } \mathcal{B}(\alpha, c) = \{x_{[m,n]} : x \in W(\alpha, c),\ m \leq n\}.$$

Since $c > 1+\alpha$ and $r(\mu) < c$, we have $\mu \in W(\alpha, c)$, so that both $W(\alpha, c)$ and $\mathcal{B}(\alpha, c)$ are not empty sets. A point $x \in \mathcal{A}^{\mathbb{N}_0}$ is *recurrent* if 0 belongs to the closure of $\{\Sigma_n(x) : n \in \mathbb{N}_0\}$.

Lemma 5.2.4. *Let $c > 1+\alpha$.*

(1) *$\mathcal{B}(\alpha, c)$ is the language of a shift space.*

(2) *For $w \in \mathcal{B}(\alpha, c)$, there is a point $y \in W(\alpha, c)$ such that y occurs in x and y is recurrent.*

Proof. (1) Suppose that $w \in \mathcal{B}(\alpha, c)$ and w occurs in $x \in W(\alpha, c)$. It is obvious that every subblock of w belongs to $\mathcal{B}(\alpha, c)$. There is a positive number N such that $\Sigma_N(\mu) < c - r(x)$. Let $y = \mu_{[1,N]}x$. It is clear that $r(y) \leq \max\{r(\mu), \Sigma_N(\mu) + r(x)\} < c$, so that $y \in W(\alpha, c)$. Thus there are nonempty block $u, v \in \mathcal{B}(\alpha, c)$ such that uwv belongs to $\mathcal{B}(\alpha, c)$. From Proposition 2.1.1(2), the statement (1) holds.

(2) Let $w \in \mathcal{B}(\alpha, c)$. There are $x \in W(\alpha, c)$, a number $N \geq 1$ and $a_1, a_2, \cdots, a_k \in \mathcal{A}$ such that w occurs in $x_{[1,N]}$ and

$$-1 - \alpha \leq \Sigma_N(x) + a_1 + \cdots + a_k \leq -1. \tag{5.6}$$

We put $y = x_{[1,N]}a_1 \cdots a_k\mu$. Then w occurs in y and

$$\begin{aligned}\{\Sigma_n(y) : n \in \mathbb{N}_0\} = \{&\Sigma_k(x) : k = 1, \cdots, N\} \\ &\cup \{\Sigma_N(x) + a_1 + \cdots + a_j : j = 1, \cdots, k\} \\ &\cup \{\Sigma_N(x) + a_1 + \cdots + a_k + \Sigma_m(\mu) : m \in \mathbb{N}_0\}.\end{aligned}$$

If $\Sigma_n(y)$ is in the first set, then $|\Sigma_n(y)| \leq r(x)$. If $\Sigma_n(y)$ is in the second set, (5.6) implies that $|\Sigma_n(y)| \leq 1+\alpha$. Since $\Sigma_m(\mu) \in [0, 1+\alpha]$, (5.6) implies that the third set is contained in $[-1-\alpha, \alpha]$. Thus we have $|\Sigma_n(y)| \leq \max\{r(x), 1+\alpha\}$ and $r(y) < c$, so that $y \in W(\alpha, c)$. From (5.6) 0 belongs to the third set, so y is recurrent. $\square$

Let X be a subshift of $\mathcal{A}^{\mathbb{Z}}$ such that $\mathcal{B}(X) = \mathcal{B}(\alpha, c)$. We will show that the following:

Theorem 5.2.5. *The shift space X is irreducible and has positive entropy, but X has no periodic points.*

Proof. To prove the irreducibility of X suppose that $u, v \in \mathcal{B}(X)$. From Lemma 5.2.4 there are two points $x, y \in W(\alpha, c)$ and a positive number $N \geq 1$ such that (i) u occurs in $x_{[1,N]}$; (ii) v occurs in y; (iii) x is recurrent; (iv) $|\Sigma_N(x)| < c - r(y)$. We put $z = x_{[1,N]}y$. It is enough to show that $r(z) < c$. Since $\Sigma_n(z) = \Sigma_j(x)$ for some $j = 1, \cdots, N$ or $\Sigma_n(z) = \Sigma_N(x) + \Sigma_i(y)$ for some $i \in \mathbb{N}_0$, we obtain $r(z) < c$.

Now we will show that $h(X) > 0$. There is a positive integer N such that $N\alpha < 1 < (N+1)\alpha$. We put $\epsilon = \max\{1 - N\alpha, (N+1)\alpha - 1\}$, then $0 < \epsilon < 1$. Let $\mathcal{U}$ be the set of blocks $u_1 \cdots u_{|u|} \in \mathcal{A}^{N+1} \cup \mathcal{A}^{N+2}$ such that $|\{i : u_i = -1\}| = 1$. For $w = w_1 w_2 \cdots w_k \in \mathcal{A}^k$, we put $\Sigma_j(w) = w_1 + \cdots + w_j (1 \leq j \leq k)$. Then $\Sigma(u) := \Sigma_{|u|}(u) \in [-\epsilon, \epsilon]$ for $u \in \mathcal{U}$. For $k = 1, 2, \cdots$, let

$$\mathcal{U}_k = \{w_1 \cdots w_k : w_j \in \mathcal{U} \text{ and } \Sigma(w_i) \neq \Sigma(w_{i+1}) \text{ for } i = 1, \cdots, k-1\}.$$

Suppose that $v \in \mathcal{U}_k$ and $(-1)\alpha^m = v'$ is a subblock of v. Then $0 \leq m \leq 2N+1$, and $-1 \leq \Sigma(v') \leq (2N+1)\alpha - 2 < 1 + \epsilon$ by the assumptions of ϵ and N. As in the construction of μ we can find a right-infinite sequence $x \in W(\alpha, c)$ which contains v. Thus $\mathcal{U}_k \subseteq \mathcal{B}(X)$. Since $|\mathcal{U}_k| \geq 2^k$, there is an integer $L \in [(N+1)k,\ (N+2)k]$ such that $|\mathcal{B}_L(X)| \geq 2^k/(k+1)$. This argument implies that there are two infinite sequence $L_1 < L_2 < L_3 < \cdots$ and $k_1 < k_2 < k_3 < \cdots$ such that $(N+1)k_j \leq L_j \leq (N+2)k_j$ and $|B_{L_j}| \geq$

$2^{k_j}/(k_j+1)$. Since

$$\frac{1}{L_j} \log |\mathcal{B}_{L_j}(X)| \geq \frac{1}{(N+2)k_j} \log \frac{2^{k_j}}{(k_j+1)},$$

we obtain $h(X) \geq \log 2/(N+2) > 0$.

Finally we show that there is no periodic points in X. To obtain a contradiction we suppose that $x = (x_{[1,n]})^\infty \in X$. Let $\beta = x_1 + x_2 + \cdots + x_n$. Then $\beta \neq 0$ so that there is a positive integer N such that $x_{[1,nN]} = (x_{[1,n]})^N$ and $|x_1 + x_2 + \cdots + x_{nN}| = N|\beta| \geq c+1$. Thus $x_{[1,nN]} \notin \mathcal{B}(\alpha, c) = \mathcal{B}(X)$. It is a contradiction. The proof is done. □

Proof of Theorem 5.2.1. Suppose that X is given in Theorem 5.2.5. There is a point $x \in X$ such that every block in $\mathcal{B}(X)$ occurs in x since X is irreducible. Analysis similar to that in the proof of positive entropy $h(X)$ shows that there is a positive integer $L > 1$ such that $h(X) > \log L/(L+1)$. We choose a number $\delta \in [1/2, 1)$ such that $\log L/(L+1) > \log(1/\delta)$. From Theorem 5.2.3 there is an infinite subset $S \subset \mathbb{N}$ such that $h(\mathbf{X}(S)) = \log(1/\delta) < h(X)$. We set

$$\mathcal{C} = \{1x_{[0,s-1]} : s \in S\}.$$

It is obvious that $\mathcal{C}$ is infinite. We define Y to be the coded system for which $\mathcal{C}$ is a code. In fact, Y is synchronized because the symbol 1 is finitary. Since S is infinite, X is a subset of Y. For $z = \langle 10^{s_i} \rangle_{i\in\mathbb{Z}} \in \mathbf{X}(S)$, let $\Phi(z) = \langle 1x_{[0,s_i-1]} \rangle_{i\in\mathbb{Z}} \in Y$. If $y \in Y$ and y is periodic, then 1 occurs infinitely often in y, so that there is an unique point $z \in \mathbf{X}(S)$ such that $y = \Phi(z)$. If $y \in Y$ and 1 does not occur in y then $y \in X$, and y is not periodic. Thus Φ is a one-to-one correspondence between $P(\mathbf{X}(S)) \setminus \{0^\infty\}$ and $P(Y)$. Hence

$$\limsup_{n\to\infty} \frac{1}{n} \log |P_n(Y)| \leq h(\mathbf{X}(S)) < h(X) \leq h(Y)$$

and Y is not periodic saturated. The proof is done. □

5.3 Some properties of $\mathbf{X}(\xi, \xi', X)$

Let $\mathcal{A}$ be a finite alphabet and Y be a shift space over $\mathcal{A}$. A point $y \in Y$ is *forwardly transitive* if the set $\{\sigma_Y^n(y) : n = 0, 1, 2, \cdots\}$ is dense in Y. Note that if $y \in Y$ is forwardly transitive and $w \in \mathcal{B}(Y)$ occurs in $y_{[0,\infty)}$ then w occurs infinitely often in $y_{[0,\infty)}$. A right-infinite sequence $\xi \in \mathcal{A}^{\mathbb{N}}$ is *transitive* if whenever a block $u \in \mathcal{A}^{|u|}$ occurs in ξ then it occurs infinitely often in ξ. We say that ξ is *eventually constant* if there is a non-negative integer N such that $\xi_n = \xi_N$ for all $n \geq N$.

Let $\mathcal{L}(\xi)$ denote the set of blocks $w \in \mathcal{A}^{|w|}$ such that w occurs infinitely often in ξ. Then $\mathcal{L}(\xi)$ is the language of a shift space, say $\mathbf{X}(\xi)$: let $w \in \mathcal{L}(\xi)$. It is obvious that every subblock of w belongs to $\mathcal{L}(\xi)$. It is easily seen that there are $u, v \in \mathcal{L}(\xi)$ so that $uwv \in \mathcal{L}(\xi)$ since $\mathcal{A}$ is finite and $w \in \mathcal{L}(\xi)$.

If ξ is transitive then $\mathbf{X}(\xi)$ is irreducible by the definition of $\mathcal{L}(\xi)$. Also, if y is forwardly transitive in Y then $Y = \mathbf{X}(y_{[0,\infty)})$ since every block occurs infinitely often in $y_{[0,\infty)}$.

Proposition 5.3.1. *Let $x \in \mathcal{A}^{\mathbb{Z}}$. The following are equivalent.*

(1) *x belongs to $\mathbf{X}(\xi)$.*

(2) *For each $n \geq 0$, $x_{[-n,n]}$ occurs infinitely often in ξ.*

(3) *For each $n \geq 0$, $x_{[-n,n]}$ occurs in ξ.*

Proof. It is enough to show that (3) implies (2). The rest proof is left to the reader. Suppose that (3) hold for $x \in \mathcal{A}^{\mathbb{Z}}$. Let $n \geq 0$, then $x_{[-n,n]}$ occurs in ξ. Suppose that i is the last coordinate of ξ so that $\xi_{[i-n,i+n]} = x_{[-n,n]}$ and $\xi_{[i'-n,i'+n]} \neq x_{[-n,n]}$ for all $i' > i$. Let $m = i - n + 1$. By (3) $x_{[-n-m,n+m]}$ occurs in ξ. Note that $x_{[-n,n]}$ is a subblock of $x_{[-n-m,n+m]}$. Since $x_{[-n,n]}$ does not occur in $\xi_{[i-n+1,\infty)}$, $x_{[-n-m,n+m]}$ must be equal to the block $\xi_{[i-n-m,i+n+m]}$. However it is impossible since $i - n - m = -1$. Thus $x_{[-n,n]}$ occurs infinitely often in ξ. $\square$

Proposition 5.3.2. *Let $\xi \in \mathcal{A}^{\mathbb{N}}$. Then $|\mathbf{X}(\xi)| = 1$ if and only if ξ is eventually constant.*

Proof. If ξ is eventually constant, then there is a number $N \geq 0$ such that $\xi_n = \xi_N = a$ for $n \geq N$. Hence $\mathcal{L}(\xi)$ is the set of a^m for all m, and $\mathbf{X}(\xi) = \{a^\infty\}$.

Suppose that, for all N, there is a positive integer $n \geq 1$ such that $\xi_{N+n} \neq \xi_N$. Then $\xi_{[N,N+n]}$ begins with $a^k b$ where $1 \leq k \leq n$, $a, b \in \mathcal{A}$ and $a \neq b$. Hence there is $j \in [N, N+n-1]$ such that $\xi_j \xi_{j+1} = ab$. We can find an infinite sequence $N_1 < N_2 < N_3 < \cdots$ such that $\xi_{N_j} \neq \xi_{N_j+1}$. Since $\mathcal{A}$ is finite, there are two different symbols $a, b \in \mathcal{A}$ such that $\xi_m \xi_{m+1} = ab$ for infinitely many $m \in \{N_1, N_2, N_3, \cdots\}$. There is a point $x \in \mathbf{X}(\xi)$ with $x_{[0,1]} = ab$. Thus $x \neq \sigma_{\mathbf{X}(\xi)}(x)$ and $|\mathbf{X}(\xi)| \geq 2$. $\square$

From now on suppose that $\mathcal{A}'$ be a finite alphabet and $\mathcal{A} \cap \mathcal{A}'$ is empty. Let $\xi' \in (\mathcal{A}')^{\mathbb{N}}$ and let X be an irreducible subshift of $\{0,1\}^{\mathbb{Z}}$. We set

$$X_0 = \left\{ x \in X \ : \ \begin{array}{l} |\{i \in (-\infty, 0] : x_i = 0\}| = \infty \ \text{ and} \\ |\{i \in (-\infty, 0] : x_i = 1\}| = \infty \end{array} \right\}.$$

Then X_0 is shift-invariant. Since both 0 and 1 occur infinitely often in all points of X_0 and X is irreducible, there are dense open subsets $G_1, G_2, G_3, \cdots$ of X such that $X_0 = \bigcap_{n \geq 1} G_n$ (i.e., X_0 is a G_δ in X). By the Baire category theorem X_0 is dense in X.

For $x \in X_0$, let $\mathcal{M}(x) = \{i \in \mathbb{Z} : x_{i-1} \neq x_i\}$. The integers in $\mathcal{M}(x)$ subdivide $\mathbb{Z}$ into intervals $I(i)$ $(i \in \mathcal{M}(x))$: if $i, j \in \mathcal{M}(x)$ and $x_k = x_i$ for $k = i+1, \cdots, j-1$, then we write $I(i) = [i, j)$. If $i \in \mathcal{M}(x)$ and $x_k = x_i$ for all $k \geq i$, then we write $I(i) = [i, \infty)$. We call $j - i$ the length of $[i, j)$ and denote the length $j - i$ by $|[i, j)|$. Since $x_k = 0$ and $x_{k'} = 1$ for infinitely many $k, k' \in (-\infty, 0]$, there are no intervals of the form $(-\infty, i)$ or $(-\infty, \infty)$. For $i \in \mathcal{M}(x)$, let $J(i)$ denote the set $\{k - i : k \in I(i)\}$: hence $J(i) = [0, j - i)$ in the case when $I(i) = [i, j)$, and $J(i) = [0, \infty)$ in the case when $I(i) = [i, \infty)$.

Remark 5.3.3. Let $x \in X_0$. Then $I(i)$, $i \in \mathcal{M}(x)$, are mutually disjoint, and $\mathbb{Z} = \bigcup_{i \in \mathcal{M}(x)} I(i)$. Let $j \in \mathcal{M}(x)$. If $I(j)$ is finite, then so is $J(j)$ and $|I(j)| = |J(j)|$. If $I(j)$ is infinite, then so is $J(j)$.

We define a map $\Phi : X_0 \to ((\mathcal{A} \cup \mathcal{A}')^{\mathbb{Z}}, \sigma)$ by

$$\Phi(x)_{I(i)} = \begin{cases} \xi_{J(i)} & \text{if } x_i = 0 \\ \xi'_{J(i)} & \text{if } x_i = 1 \end{cases} \qquad (x \in X_0,\ i \in \mathcal{M}(x)).$$

By Remark 5.3.3 Φ is well-defined. It is clear that Φ is continuous on X_0. It is easy to show that $\Phi(\sigma_X(x)) = \sigma(\Phi(x))$ for $x \in X_0$. Thus $\Phi(X_0)$ is a shift-invariant subset of $(\mathcal{A} \cup \mathcal{A}')^{\mathbb{Z}}$, and the closure of $\Phi(X_0)$ is a subshift of $(\mathcal{A} \cup \mathcal{A}')^{\mathbb{Z}}$. We denote the closure of $\Phi(X_0)$ by $\mathbf{X}(\xi, \xi', X)$. Note that $y \in \mathbf{X}(\xi, \xi', X)$ if and only if for all $n \geq 0$ there is a point $x \in X_0$ such that $y_{[-n,n]} = \Phi(x)_{[-n,n]}$.

Proposition 5.3.4. *$\mathbf{X}(\xi, \xi', X)$ is irreducible.*

Proof. Suppose that u and v are allowed blocks in $\mathbf{X}(\xi, \xi', X)$. There are two points $x, y \in X_0$ such that $u = \Phi(x)_{[0,|u|)}$ and $v = \Phi(y)_{[0,|v|)}$. Since $x, y \in X$, there are $i, j < 0$ such that

$$x_i \neq x_0, \quad y_j \neq y_0, \quad x_{[i+1,x_0]} = (x_0)^{-i} \text{ and } y_{[j+1,y_0]} = (y_0)^{-j}. \tag{5.7}$$

There is a non-empty block $w \in \mathcal{B}(X)$ such that $x_{[i,|u|)} w y_{[j,|v|)} \in \mathcal{B}(X)$ since X is irreducible. By the compactness and the irreducibility of X, there is a point $z \in X_0$ such that $z_{[i,\infty)}$ begins with $x_{[i,|u|)} w y_{[j,|v|)}$ and $\Phi(z) \in \mathbf{X}(\xi, \xi', X)$. Let l be an integer with $z_{[l-|v|,l-j)} = y_{[j,|v|)}$. (5.7) implies that $\Phi(z)_{[i,l-j)} = u' u w' v$ is allowed in $\mathbf{X}(\xi, \xi', X)$, and the proof is done. □

If a point in $\mathbf{X}(\xi, \xi', X)$ consists of symbols in $\mathcal{A}$(or $\mathcal{A}'$, respectively), then the point belongs to $\mathbf{X}(\xi)$(or $\mathbf{X}(\xi')$, respectively).

Lemma 5.3.5. *$\mathbf{X}(\xi, \xi', X) \cap \mathcal{A}^{\mathbb{Z}} \subseteq \mathbf{X}(\xi)$ and $\mathbf{X}(\xi, \xi', X) \cap (\mathcal{A}')^{\mathbb{Z}} \subseteq \mathbf{X}(\xi')$.*

Proof. The proof is an immediate consequence of Proposition 5.3.1. □

We define a map $\Psi : \mathbf{X}(\xi, \xi', X) \to \{0,1\}^{\mathbb{Z}}$ by

$$\Psi(y)_i = \begin{cases} 0 & \text{if } y_i \in \mathcal{A} \\ 1 & \text{if } y_i \in \mathcal{A}' \end{cases} \qquad (y \in \mathbf{X}(\xi, \xi', X),\ i \in \mathbb{Z}).$$

It is obvious that Ψ is continuous on $\mathbf{X}(\xi, \xi', X)$, and that whenever $x \in X_0$ then $\Psi(\Phi(x)) = x$. Thus Ψ is one-to-one on $\Phi(X_0)$, and Φ is one-to-one on X_0.

Lemma 5.3.6. $\Psi(\mathbf{X}(\xi, \xi', X)) = X$.

Proof. Since Ψ is continuous and $\Psi(\Phi(X_0)) = X_0$ and $\overline{X_0} = X$, the equality hold. □

We consider two points 0^∞ and 1^∞. Then (i) both 0^∞ and 1^∞ belong to X; (ii) neither of them belong to X; (iii) one of them belongs to X. In the case when (ii) hold, $X_0 = X$. Indeed, there is a positive number m such that $0^m \notin \mathcal{B}(X)$ and $1^m \notin \mathcal{B}(X)$. Thus, if $x \in X$, then both 0 and 1 occur infinitely often in $x_{(-\infty,0]}$. Hence $x \in X_0$.

Proposition 5.3.7. *The following are equivalent:*

(1) $0^\infty \in X$.

(2) $\mathbf{X}(\xi) \subseteq \mathbf{X}(\xi, \xi', X)$.

(3) $\mathbf{X}(\xi, \xi', X) \cap \mathcal{A}^{\mathbb{Z}} \neq \emptyset$.

In this case, $\mathbf{X}(\xi, \xi', X) \cap \mathcal{A}^{\mathbb{Z}} = \mathbf{X}(\xi)$. *Similarly, the following are equivalent:*

$(1)'$ $1^\infty \in X$.

$(2)'$ $\mathbf{X}(\xi') \subseteq \mathbf{X}(\xi, \xi', X)$.

$(3)'$ $\mathbf{X}(\xi, \xi', X) \cap (\mathcal{A}')^{\mathbb{Z}} \neq \emptyset$.

In this case, $\mathbf{X}(\xi, \xi', X) \cap (\mathcal{A}')^{\mathbb{Z}} = \mathbf{X}(\xi')$.

Proof. We only show that the equivalence of (1), (2) and (3). By Lemma 5.3.5, $\mathbf{X}(\xi, \xi', X) \cap \mathcal{A}^{\mathbb{Z}} = \mathbf{X}(\xi)$.

Suppose that (1) hold. Let $y \in \mathbf{X}(\xi)$ and $n \in \mathbb{N}$. By Proposition 5.3.1, $y_{[-n,n]}$ occurs in ξ. Let $x \in X_0$ such that $x_{[i,\infty)}$ consists of only the symbol 0 and $x_{i-1} = 1$ for some $i \leq 0$. Then $\Phi(x)_{[i,\infty)} = \xi$, and $y_{[-n,n]}$ occurs in $\Phi(x) \in \mathbf{X}(\xi, \xi', X)$. Since n is arbitrary, $y \in \mathbf{X}(\xi, \xi', X)$ and (2) hold.

We show that (2) $\Rightarrow$ (3). Since $\mathcal{A}$ is finite, $\mathcal{L}(\xi) \neq \emptyset$. Thus $\mathbf{X}(\xi)$ is a non-empty set. From (2) and the inclusion relation $\mathbf{X}(\xi) \subseteq \mathcal{A}^{\mathbb{Z}}$, (3) hold.

If $y \in \mathbf{X}(\xi, \xi', X) \cap \mathcal{A}^{\mathbb{Z}}$, then $\Psi(y) = 0^\infty$. Lemma 5.3.6 implies that $0^\infty \in X$, so that (3) implies that (1). □

Proposition 5.3.8. *The following are equivalent:*

(1) Φ *is uniformly continuous on* X_0*.*

(2) $\Psi : \mathbf{X}(\xi, \xi', X) \to X$ *is a conjugacy.*

(3) *If* $0^\infty \in X$ *then* $|\mathbf{X}(\xi)| = 1$*, and if* $1^\infty \in X$ *then* $|\mathbf{X}(\xi')| = 1$*.*

Proof. Suppose that (1) hold. Since X_0 is dense in X, there is a continuous extension Φ' of Φ from X to $\mathbf{X}(\xi, \xi', X)$ such that the restriction of Φ' on X_0 is Φ. It is easily seen that Φ' is the inverse function of Ψ, and (2) hold.

Suppose that (2) hold. If $0^\infty \in X$, Proposition 5.3.7(3) implies that $\Psi(\mathbf{X}(\xi)) = \{0^\infty\}$. Since Ψ is a conjugacy, $|\mathbf{X}(\xi)| = 1$. Similarly if $1^\infty \in X$ then $|\mathbf{X}(\xi')| = 1$.

Finally we show that (3) $\Rightarrow$ (1). We only consider the case when $0^\infty \in X$ and $1^\infty \notin X$. The proof of the other cases is left to the reader. By (3), $|\mathbf{X}(\xi)| = 1$. From Proposition 5.3.2 there is an integer N such that $\xi_n = \xi_N$ for $n \geq N$. We can choose a positive integer M so that $1^M \notin \mathcal{B}(X)$ since $1^\infty \notin X$. Let $m = \max\{N, M-1\}$. We will show that if $x \in X_0$ then $x_{[-m,0]}$ determines the value $\Phi(x)_0$. If $x_0 = 1$, there is the largest number $i \in [-m, -1]$ such that $x_i = 0$ and $x_{i+1} = 1$. Then $\Phi(x)_0 = (\xi')_{-i-1}$. Suppose that $x_0 = 0$. If $x_{[-m,0]} = 0^{m+1}$ then $\Phi(x)_0$ occurs in $\xi_{[N,\infty)}$ since $m+1 > N$. Thus $\Phi(x)_0 = \xi_N$. If there is a number $i \in [-m+1, 0]$ such that $x_{i-1} = 1$ but $x_j = 0$ $(i \leq j \leq 0)$, then $\Phi(x)_0 = \xi_{-i}$. □

To compute the zeta function of $\mathbf{X}(\xi, \xi', X)$ we need to know the number of periodic points in $\mathbf{X}(\xi, \xi', X)$. Recall that $P(S)$ is the set of periodic points in S.

Proposition 5.3.9. *If both* 0^∞ *and* 1^∞ *are in* X*, then*

$$P(\mathbf{X}(\xi, \xi', X)) = P(\mathbf{X}(\xi)) \cup P(\mathbf{X}(\xi')) \cup \Phi(P(X) \setminus \{0^\infty, 1^\infty\})$$

and the union is disjoint.

Proof. Suppose that both 0^∞ and 1^∞ belong to X. The union is disjoint since $\mathcal{A}\cap\mathcal{A}'$ is empty. From Proposition 5.3.7, $\mathbf{X}(\xi),\mathbf{X}(\xi')\subseteq\mathbf{X}(\xi,\xi',X)$; hence $P(\mathbf{X}(\xi))\cup P(\mathbf{X}(\xi'))\subseteq P(\mathbf{X}(\xi,\xi',X))$. Since $P(X)\setminus\{0^\infty,1^\infty\}\subseteq X_0$, we have $\Phi(P(X)\setminus\{0^\infty,1^\infty\})\subseteq P(\mathbf{X}(\xi,\xi',X))$. The Part($\supseteq$) is proved.

Conversely, let $y\in P(\mathbf{X}(\xi,\xi',X))$. If y is in either $\mathcal{A}^{\mathbb{Z}}$ or $(\mathcal{A}')^{\mathbb{Z}}$, then $y\in P(\mathbf{X}(\xi))\cup P(\mathbf{X}(\xi'))$. If both $a\in\mathcal{A}$ and $b\in\mathcal{A}'$ occur in y, then $\Psi(y)$ is periodic in X by Lemma 5.3.6, and $\Psi(y)\neq\{0^\infty,1^\infty\}$; hence $\Psi(y)\in P(X)\setminus\{0^\infty,1^\infty\}$. Since $\Phi(\Psi(y))=y$, $y\in\Phi(P(X)\setminus\{0^\infty,1^\infty\})$. The Part($\subseteq$) is proved. $\square$

Combining the definition of the zeta function with Proposition 5.3.9 yields the following theorem.

Theorem 5.3.10. *The zeta function of $\mathbf{X}(\xi,\xi',X)$ is of the form:*

$$\zeta_{\mathbf{X}(\xi,\xi',X)}(t)=\zeta_{\mathbf{X}(\xi)}(t)\zeta_{\mathbf{X}(\xi')}(t)\zeta_X(t)(1-t)^2.$$

Remark 5.3.11. If neither 0^∞ nor 1^∞ is in X, then $P(\mathbf{X}(\xi,\xi',X))=\Phi(P(X))$ by Proposition 5.3.9. Thus $\zeta_{\mathbf{X}(\xi,\xi',X)}(t)=\zeta_X(t)$. Similarly, if $0^\infty\in X$ and $1^\infty\notin X$ then $\zeta_{\mathbf{X}(\xi,\xi',X)}(t)=\zeta_{\mathbf{X}(\xi)}(t)\zeta_X(t)(1-t)$, and if $0^\infty\notin X$ and $1^\infty\in X$ then $\zeta_{\mathbf{X}(\xi,\xi',X)}(t)=\zeta_{\mathbf{X}(\xi')}(t)\zeta_X(t)(1-t)$.

Before we compute the entropy of $\mathbf{X}(\xi,\xi',X)$, we define the entropy of an infinite subset $\mathcal{L}$ of $\bigcup_{m=0}^\infty\mathfrak{A}^m$ for a finite alphabet $\mathfrak{A}$.

Suppose that $\mathfrak{A}$ is a finite alphabet. A subset $\mathcal{L}$ of $\bigcup_{m=0}^\infty\mathfrak{A}^m$ is said to *have the subblock property* if whenever $w\in\mathcal{L}$ then every subblock of w belongs to $\mathcal{L}$. For example, both $\mathcal{L}(\xi)$ and $\mathcal{L}(\xi')$ have the subblock property. It is obvious that $\mathcal{B}(Y)$ has the subblock property for a shift space Y.

From now on let $\mathcal{L}$ be a subset of $\bigcup_{m=0}^\infty\mathfrak{A}^m$.

Lemma 5.3.12. *If $\mathcal{L}$ has the subblock property, then*

$$|\mathcal{L}\cap\mathfrak{A}^{m+n}|\leq|\mathcal{L}\cap\mathfrak{A}^m||\mathcal{L}\cap\mathfrak{A}^n|$$

for all $m,n\in\mathbb{N}$.

Proof. Let $m, n \in \mathbb{N}$. The map

$$\mathcal{L} \cap \mathfrak{A}^{m+n} \ni w = w_0 \cdots w_{m+n-1} \mapsto (w_{[0,m)}, w_{[m,m+n)}) \in (\mathcal{L} \cap \mathfrak{A}^m) \cup (\mathcal{L} \cap \mathfrak{A}^n)$$

is well-defined by the subbblock property. Also, this map is one-to-one, so we obtain the desired result. □

If $\mathcal{L}$ is finite, then $\mathcal{L} \cap \mathfrak{A}^n = \emptyset$ for sufficiently large n. If $\mathcal{L}$ is infinite, we obtain the following proposition by using Lemma 5.3.12. The proof is same to that of [LinM, Proposition 4.1.8]. The details are left to the reader.

Proposition 5.3.13. *If $\mathcal{L}$ has the subblock property and $|\mathcal{L}| = \infty$, then*

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L} \cap \mathfrak{A}^n| = \inf_{n \geq 1} \left\{ \frac{1}{n} \log |\mathcal{L} \cap \mathfrak{A}^n| \right\}.$$

Definition 5.3.14. If $\mathcal{L}$ is infinite and has the subblock property, then *the entropy $h(\mathcal{L})$ of $\mathcal{L}$* is defined by $\lim_{n\to\infty}(1/n) \log |\mathcal{L} \cap \mathfrak{A}^n|$.

If Y is a shift space, then $\mathcal{B}(Y)$ is infinite and has the subblock property, hence $h(X) = h(\mathcal{B}(X))$.

Proposition 5.3.15. *Suppose that $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \cdots$ are subsets of $\bigcup_{m=0}^{\infty} \mathfrak{A}^m$. If $\mathcal{L}_1 \supseteq \mathcal{L}_2 \supseteq \mathcal{L}_3 \supseteq \cdots$, each $\mathcal{L}_n$ is infinite and each $\mathcal{L}_n$ has the subblock property, then $\bigcap_{n=1}^{\infty} \mathcal{L}_n$ is infinite and has the subblock property and $h(\bigcap_{n=1}^{\infty} \mathcal{L}_n) = \lim_{m\to\infty} h(\mathcal{L}_m)$.*

Proof. It is obvious that $\bigcap_{n=1}^{\infty} \mathcal{L}_n$ has the subblock property. To show that $\bigcap_{n=1}^{\infty} \mathcal{L}_n$ is infinite, let $k \geq 1$ be arbitrary. We consider $\langle \mathcal{L}_n \cap \mathfrak{A}^k \rangle_{n=1}^{\infty}$. Since each $\mathcal{L}_n$ is infinite and has the subblock property, we have $\mathcal{L}_n \cap \mathfrak{A}^k \neq \emptyset$ for all n. Since $\mathcal{L}_n \supseteq \mathcal{L}_{n+1}$, we have $\mathcal{L}_n \cap \mathfrak{A}^k \supseteq \mathcal{L}_{n+1} \cap \mathfrak{A}^k$. Since $\mathfrak{A}^k$ is finite, so is $\mathcal{L}_n \cap \mathfrak{A}^k$, $n \geq 1$. Thus there is a positive number M such that

$$\mathcal{L}_m \cap \mathfrak{A}^k = \mathcal{L}_M \cap \mathfrak{A}^k \neq \emptyset \quad (m \geq M).$$

Thus $(\bigcap_{n=1}^{\infty} \mathcal{L}_n) \cap \mathfrak{A}^k = \mathcal{L}_M \cap \mathfrak{A}^k$ is non-empty. Since k is arbitrary, $\bigcap_{n=1}^{\infty} \mathcal{L}_n$ is infinite.

The proof of the entropy of $\bigcap_{n=1}^{\infty} \mathcal{L}_n$ is same to that of [LinM, Proposition 4.4.6]. The details are left to the reader. $\square$

Let $\gamma \in \mathfrak{A}^{\mathbb{N}}$. Recall that $\mathcal{L}(\gamma)$ is the set of blocks $w \in \mathfrak{A}^{|w|}$ such that w occurs infinitely often in γ. Since $\mathcal{L}(\gamma)$ is the language of $\mathbf{X}(\gamma)$, we have $h(\mathbf{X}(\gamma)) = h(\mathcal{L}(\gamma))$. Let $\mathcal{L}'(\gamma)$ denote the set of blocks $u \in \mathfrak{A}^{|u|}$ such that u occurs in γ. It is clear that $\mathcal{L}'(\gamma)$ has the subblock property, and that $\mathcal{L}(\gamma)$ is a subset of $\mathcal{L}'(\gamma)$. By definition, both $\mathcal{L}(\gamma)$ and $\mathcal{L}'(\gamma)$ are infinite. Thus $h(\mathcal{L}'(\gamma)) \geq h(\mathcal{L}(\gamma)) = h(\mathbf{X}(\gamma))$. The following theorem shows that $h(\mathbf{X}(\gamma)) = h(\mathcal{L}'(\gamma))$.

Theorem 5.3.16. *$h(\mathcal{L}'(\gamma)) = h(\mathcal{L}(\gamma))$.*

Proof. Let $k = 1, 2, 3, \cdots$ and let $\gamma = \gamma_0\gamma_1\gamma_2\cdots \in \mathfrak{A}^{\mathbb{N}}$. We put $\sigma^k\gamma = \gamma_k\gamma_{k+1}\gamma_{k+2}\cdots \in \mathfrak{A}^{\mathbb{N}}$. Then the following (i), (ii), (iii) and (iv) are immediate consequences of definitions:

(i) Each $\mathcal{L}'(\sigma^k\gamma)$ is infinite,

(ii) Each $\mathcal{L}'(\sigma^k\gamma)$ has the subblock property,

(iii) $\mathcal{L}'(\sigma^k\gamma) \supseteq \mathcal{L}'(\sigma^{k+1}\gamma)$,

(iv) $h(\mathcal{L}'(\sigma^k\gamma)) \geq h(\mathcal{L}'(\sigma^{k+1}\gamma))$.

For each $n = 1, 2, \cdots$, we have

$$\left(\mathcal{L}'(\sigma^{k+1}\gamma) \cap \mathfrak{A}^n\right) \cup \{\gamma_{[k,k+n)}\} = \mathcal{L}'(\sigma^k\gamma) \cap \mathfrak{A}^n.$$

Then

$$\begin{aligned}\frac{1}{n}\log|\mathcal{L}'(\sigma^k\gamma) \cap \mathfrak{A}^n| &= \frac{1}{n}\log(|\mathcal{L}'(\sigma^{k+1}\gamma) \cap \mathfrak{A}^n| + 1) \\ &\leq \frac{1}{n}\log(2\,|\mathcal{L}'(\sigma^{k+1}\gamma) \cap \mathfrak{A}^n|)\end{aligned}$$

so that $h(\mathcal{L}'(\sigma^k\gamma)) \leq h(\mathcal{L}'(\sigma^{k+1}\gamma))$, and $h(\mathcal{L}'(\sigma^k\gamma)) = h(\mathcal{L}'(\sigma^{k+1}\gamma))$ by (iv). We obtain

(v) $h(\mathcal{L}'(\gamma)) = h(\mathcal{L}'(\sigma^k\gamma))$ for all k.

We have $\mathcal{L}(\gamma) = \bigcap_{k=1}^{\infty} \mathcal{L}'(\sigma^k\gamma)$ since $\sigma^k\gamma = \gamma_{[k,\infty)}$. From (i), (ii), (iii), (v) and Proposition 5.3.15, we have

$$h(\mathcal{L}(\gamma)) = \lim_{n\to\infty} h(\mathcal{L}'(\sigma^k\gamma)) = h(\mathcal{L}'(\gamma)).$$

The proof is done. $\square$

Now we compute the entropy of $\mathbf{X}(\xi, \xi', X)$.

Theorem 5.3.17. *If both 0^∞ and 1^∞ are in X, then*

$$h(\mathbf{X}(\xi, \xi', X)) = \max\{h(\mathbf{X}(\xi)), h(\mathbf{X}(\xi')), h(X)\}. \tag{5.8}$$

Proof. Suppose that $0^\infty, 1^\infty \in X$. Let H be the right-hand side of (5.8). By Lemma 5.3.6 and Proposition 5.3.7, $h(\mathbf{X}(\xi, \xi', X)) \geq H$. To prove that $h(\mathbf{X}(\xi, \xi', X)) \leq H$, we suppose that $\log\lambda > H$ and $\lambda > 1$. Theorem 5.3.16 implies that

$$H = \max\{h(\mathcal{L}'(\xi)), h(\mathcal{L}'(\xi')), h(X)\}.$$

Hence there is a constant $c \geq 1$ such that

$$|\mathcal{L}'(\xi) \cap \mathcal{A}^n|,\ |\mathcal{L}'(\xi') \cap (\mathcal{A}')^n|,\ |\mathcal{B}_n(X)|\ \leq\ c\lambda^n \qquad (n = 1, 2, \cdots).$$

Let $n \geq 1$. If we put

$$\begin{aligned} A_n &= \mathcal{B}_n(\mathbf{X}(\xi, \xi', X)) \cap \mathcal{A}^n, \qquad B_n = \mathcal{B}_n(\mathbf{X}(\xi, \xi', X)) \cap (\mathcal{A}')^n \\ C_n &= \mathcal{B}_n(\mathbf{X}(\xi, \xi', X)) \setminus (A_n \cup B_n), \end{aligned}$$

then $\mathcal{B}_n(\mathbf{X}(\xi, \xi', X)) = A_n \cup B_n \cup C_n$ and the union is disjoint. It is clear that $A_n \subseteq \mathcal{L}'(\xi) \cap \mathcal{A}^n$ and $B_n \subseteq \mathcal{L}'(\xi') \cap (\mathcal{A}')^n$; hence

$$|A_n| \leq c\lambda^n \quad \text{and} \quad |B_n| \leq c\lambda^n \tag{5.9}$$

Suppose that $w = w_0 w_1 \cdots w_{n-1} \in C_n$. Let

$$m = m(w) = \min\{j \in [1, n-1] : \chi_{\mathcal{A}'}(w_{j-1}) = \chi_{\mathcal{A}}(w_j)\}$$

where χ_S is the characteristic function on a set S. Then $w_{[0,m)}$ is in either A_n or B_n. Also, we have

$$\chi_{\mathcal{A}'}(w_{m-1})\chi_{\mathcal{A}'}(w_m) \cdots \chi_{\mathcal{A}'}(w_{n-1}) \in \mathcal{B}_{n-m+1}(X).$$

We denote this block briefly by $\chi(w_{[m-1,n)})$. We claim that w is completely determined by $w_{[0,m)}$ and $\chi(w_{[m-1,n)})$. Suppose that $x \in X_0$ and $x_{[m-1,n)} = \chi(w_{[m-1,n)})$. Since

$$x_{m-1} = \chi_{\mathcal{A}'}(w_{m-1}) = \chi_{\mathcal{A}}(w_m) \neq \chi_{\mathcal{A}'}(w_m) = x_m,$$

we have $\Phi(x)_{[m-1,n)} = w_{[m,n)}$. Thus whenever $u \in C_n$, $u_{[0,m)} = w_{[0,m)}$ and $\chi(u_{[m-1,n)}) = \chi(u_{[m-1,n)})$, then we obtain $u = w$ and the claim is proved. By (5.9)

$$|C_n| \leq \sum_{k=1}^{n-1} \big(|A_k \cup B_k||\mathcal{B}_{n-k+1}(X)|\big) \;\leq\; 2c^2(n-1)\lambda^{n+1}.$$

Combining this $|C_n|$ with (5.9) yields

$$|\mathcal{B}_n(\mathbf{X}(\xi, \xi', X))| = |A_n| + |B_n| + |C_n| \leq 2c\lambda^n(1 - c\lambda + c\lambda n).$$

We have $(1/n)\log(2c\lambda^n(1 - c\lambda + c\lambda n)) \to \log\lambda$ as $n \to 0$ since $c\lambda > 1$ and $\lim_{n\to\infty} n^{1/n} = 1$. Therefore $h(\mathbf{X}(\xi, \xi', X)) \leq \log\lambda$. Since $\log\lambda$ is arbitrary, $h(\mathbf{X}(\xi, \xi', X)) = H$ and the proof is done. $\square$

We conclude this section by examples.

Examples 5.3.18. (1) The simplest $\mathbf{X}(\xi, \xi', X)$ may be $\mathbf{X}(0^\infty, 1^\infty, \{0,1\}^{\mathbb{Z}})$. Note that this shift is the full $\{0,1\}$-shift.

(2) Suppose that Y is the S-gap shift, Z is an irreducible subshift of $\{-1, \alpha\}^{\mathbb{Z}}$ and $z \in Z$ such that Y, Z are given in the proof of Theorem 5.2.1 and every allowed block in Z occurs in z. Also there is the coded system $\mathbf{X}(\mathcal{C})$

which is obtained from Y, Z and z. We put

$$\xi = z_{[0,\infty)} \in \{-1, \alpha\}^{\mathbb{N}}, \quad \xi' = 111\cdots = \{1\}^{\mathbb{N}} \quad \text{and} \quad X = Y.$$

Then $\mathbf{X}(\mathcal{C}) = \mathbf{X}(\xi, \xi', Y)$.

5.4 An almost sofic shift which is not periodic points dense

In this section we will show that there is an almost sofic shift which is not periodic points dense by using the result in Section 5.3.

Theorem 5.4.1. *There is an almost sofic shift which is not periodic points dense.*

Let $\mathcal{A} = \{a, b\}$ and $\mathcal{A}' = \{c, d\}$ with $\mathcal{A} \cap \mathcal{A}' = \emptyset$. Suppose that y is a forwardly transitive point in $\mathcal{A}^{\mathbb{Z}}$ and $\xi = y_{[0,\infty)}$, then $\mathbf{X}(\xi)$ is the full 2-shift $\mathcal{A}^{\mathbb{Z}}$. Let $z \in (\mathcal{A}')^{\mathbb{Z}}$ be the Morse sequence (see Section 6.3) and $\xi' = z_{[0,\infty)}$. Then $\mathbf{X}(\xi')$ is the Morse shift with entropy 0 and $P(\mathbf{X}(\xi')) = \emptyset$. We need the following theorem.

Theorem 5.4.2. *There is an irreducible subshift X of $\{0,1\}^{\mathbb{Z}}$ such that $h(X) = 0$ and $0^\infty, 1^\infty \in X$ and $P(X) = \{0^\infty, 1^\infty\}$.*

Let X be given in Theorem 5.4.2. Then we obtain an irreducible subshift $Y := \mathbf{X}(\xi, \xi', X)$ of $\{a, b, c, d\}^{\mathbb{Z}}$ as in Section 5.3. Theorem 5.3.17 implies that $h(Y) = h(\mathbf{X}(\xi)) = \log 2$. By Proposition 5.3.7, $\mathbf{X}(\xi) \subseteq Y$ and so that Y is almost sofic. Proposition 5.3.9 implies that we have $P(Y) = P(\mathbf{X}(\xi))$. Since $\mathbf{X}(\xi') \subseteq Y$(Proposition 5.3.7), we obtain Y is not periodic points dense. This completes the proof of Theorem 5.4.1.

Now it remains to show that Theorem 5.4.2 holds. We define inductively blocks $A_1, A_2, A_3, \cdots$ over $\{0,1\}$ by $A_1 = 01$ and $A_n = A_{n-1}A_{n-1}0^{|A_{n-1}|}1^{|A_{n-1}|}$ for $n \geq 2$. It is clear that A_n is a prefix of A_{n+1} and $|A_n| = 2^{2n-1}$ for $n \geq 1$.

Let $\mathcal{W}$ be the set of all blocks w over $\{0,1\}$ such that w occurs in A_n for some $n \geq 1$. Then it is clear that $\mathcal{W}$ is the language of a shift space, say X. Since A_n is a prefix of A_{n+1}, X is irreducible. Since each A_{n+1} ends with $0^{|A_n|}1^{|A_n|}$ and $\langle |A_n| \rangle$ is increasing, for each $k \geq 1$, there is a positive integer n such that 0^k and 1^k occur in A_n. Hence we obtain the following proposition.

Proposition 5.4.3. *0^∞ and 1^∞ are in X.*

We will show that X has only two periodic points 0^∞ and 1^∞(Corollary 5.4.5).

Proposition 5.4.4. *Let $n \geq 1$ and $x \in X$. Then 0^n or 1^n occurs in x.*

Corollary 5.4.5. $P(X) = \{0^\infty, 1^\infty\}$.

Proof. Suppose that $x \in P(X) \setminus \{0^\infty, 1^\infty\}$ and $\sigma_X^p(x) = x$. Then x contains both 0 and 1. If $n > p$ then both 0^n and 1^n do not occur in x since x has period p. It contradicts to Proposition 5.4.4. □

To prove that Proposition 5.4.4 we need two lemmas. The proofs are left to the reader.

Lemma 5.4.6. *Let $l \geq 1$ and $w_1, w_2, \cdots, w_8 \in \{0,1\}^l$. If u is a subblock of $w_1 w_2 \cdots w_8$ and $|u| = 4l - 1$, then at least one of the blocks in $\{w_3, w_4, w_7, w_8\}$ occurs in u.*

Lemma 5.4.7. *Let $n \geq 1$. For each $k \geq 1$, A_{n+k} is a concatenation of blocks of the form $uvab$ where $|u| = |v| = |A_n|$ and $a, b \in \{0^{|A_n|}, 1^{|A_n|}\}$.*

Proof of Proposition 5.4.4. Suppose that $n \geq 1$ and $x \in X$. Let $l = |A_n|$ and $w = x_{[1,4l-1]}$. There is a positive integer k such that w is a subblock of A_{n+k}. If $k = 1$, by definition of A_j, $0^{|A_n|}$ or $1^{|A_n|}$ occurs in w. Since $n < |A_n|$, the proof is done.

Suppose that $k \geq 2$. Since $|w| = 4l - 1$, Lemma 5.4.7 implies that w occurs in (i) $uvab$ or (ii) $uvabu'v'a'b'$, where these blocks satisfy the conditions in Lemma 5.4.7: if the case (i) holds, then 0^n or 1^n occurs in w as in the case when $k = 1$. If the case (ii) holds, then at least one of the blocks in $\{a, b, a', b'\}$ occurs in w by Lemma 5.4.6. Hence 0^n or 1^n occurs in w. The proof is done. □

Finally we prove that the topological entropy of X is 0.

Proposition 5.4.8. $h(X) = 0$.

Proof. Let $n \geq 1$ and $L = |A_n|$. Observe that every block $w \in \mathcal{B}_L(X)$ is a subblock of a block in $\mathcal{U}_n$:

$$\mathcal{U}_n = \{A_n A_n,\ A_n 0^L,\ 1^L A_n,\ 0^L 1^L,\ 1^L 0^L,\ 0^{2L},\ 1^{2L}\}.$$

Hence $|\mathcal{B}_L(X)| \leq 7(|A_n| + 1) = 7(L+1)$. For infinitely many m, we have $|\mathcal{B}_m(X)| \leq 7(m+1)$. Therefore $\liminf_{n\to\infty}(1/n)\log|\mathcal{B}_n(X)| = 0$. Since $\langle(1/n)\log|\mathcal{B}_n(X)|\rangle$ converges, $h(X) = 0$. □

The proof of Theorem 5.4.2 is completed by Propositions 5.4.3 and 5.4.8 and Corollary 5.4.5.

Chapter 6

Shift-flip systems

In this chapter, we are interested in the following property:

(P) *If (X, σ_X) has a flip, then it has infinitely many non-conjugate ones.*

In Section 6.1, we prove that every infinite sychronized system satisfies the property (P). Since every sychronized system is coded, it is a natural question whether or not a coded system has the above property. Section 6.2 shows that there is a coded system X which has finitely many non-conjugate flips. We up to now considered large spaces: they have periodic orbits and proper subshifts. In the last section we consider the Morse shift. It is an infinite minimal shift, i.e., it can not have a proper subshift. Also, it can not have periodic points. The Morse shift has finitely many non-conjuagte flips. This chapter is based on [ChoK].

6.1 Flips for a synchronized system

The goal of this section is the following:

Theorem 6.1.1. *If X is a synchronized system, $|X| = \infty$ and there is a flip for (X, σ_X), then there are infinitely many non-conjugate flips for (X, σ_X).*

Suppose that (X, σ_X, φ) is a shift-flip system. Let $A(\varphi)$ denote the set of points $x \in X$ such that a finitary block occurs infinitely often in x and

$$0 < |\{i \in \mathbb{Z} : \varphi(x)_i \neq x_i\}| < \infty.$$

Recall that $F(\varphi; n) = \{x \in X : \sigma_X^n(x) = \varphi(x) = x\}$ for $n = 1, 2, \cdots$ (Section 2.3). If (Y, σ_Y, ψ) is a shift-flip systems and Φ is a conjugacy from (X, σ_X, φ) to (Y, σ_Y, ψ), it is clear that

$$\begin{aligned} \Phi(F(\varphi; n)) &= F(\psi; n) \quad (n = 1, 2, \cdots), \\ \Phi(A(\varphi)) &= A(\psi). \end{aligned}$$

Proposition 6.1.2. *If X is an infinite synchronized system and φ is a flip for (X, σ_X), then at least one of the two sets $A(\varphi)$ and $A(\sigma_X \varphi)$ is non-empty.*

Proposition 6.1.3. *If X is an infinite synchronized system, φ is a flip for (X, σ_X), and $A(\varphi) \neq \emptyset$, then there is a flip ψ for (X, σ_X) such that*

(1) $|F(\varphi; n)| \leq |F(\psi; n)|$ *for all n,*

(2) $|F(\varphi; n)| < |F(\psi; n)|$ *for some n, and*

(3) $A(\psi) \neq \emptyset$.

These propositions are proved in the next section. We assume that Proposition 6.1.2 and 6.1.3 are proved, and prove Theorem 6.1.1.

Proof of Theorem 6.1.1. Suppose that X is an infinite synchronized system, and that φ_1 is a flip for (X, σ_X). We may assume that $A(\varphi_1) \neq \emptyset$ from Proposition 6.1.2. Applying Proposition 6.1.3 to (X, σ_X, φ_1) we obtain a shift-flip system (X, σ_X, φ_2) which satisfies conditions (1), (2), (3) of Proposition 6.1.3. We now proceed by induction. Let $n \geq 3$. Suppose that (X, σ_X, φ_n) satisfie Proposition 6.1.3. Since $A(\varphi_n)$ is non-empty, we can apply Proposition 6.1.3 to (X, σ_X, φ_n), and we obtain new shift-flip system $(X, \sigma_X, \varphi_{n+1})$. Then there is an infinite sequecne $\varphi_1, \varphi_2, \varphi_3, \cdots$ of flips for (X, σ_X) such that

(i) $|F(\varphi_i; n)| \leq |F(\varphi_{i+1}; n)|$ for all n,

(ii) $|F(\varphi_i; n)| < |F(\varphi_{i+1}; n)|$ for some n.

We show that two flips φ_i and φ_j are not conjugate. We may assume that $i < j$. By (ii) there is a positive number m with $|F(\varphi_i; m)| < |F(\varphi_{i+1}; m)|$. For this number m

$$|F(\varphi_{i+1}; m)| \leq |F(\varphi_{i+2}; m)| \leq \cdots \leq |F(\varphi_j; m)|$$

from (i). Thus $|F(\varphi_i; m)| \neq |F(\varphi_j; m)|$, and φ_i and φ_j are not conjugate. □

6.1.1 Proofs of Propositions 6.1.2 and 6.1.3

In this section we prove Propositions 6.1.2 and 6.1.3. We start with some preliminaries. Let (X, σ_X, φ) be a shift-flip system. If $\varphi(x)_0 = \varphi(x')_0$ whenever $x_0 = x'_0$, then there is a unique map $\tau : \mathcal{B}_1(X) \to \mathcal{B}_1(X)$ such that

$$\varphi(x)_i = \tau(x_{-i}) \quad (x \in X, i \in \mathbb{Z}),$$

and consequently $\tau^2 = \mathrm{id}_{\mathcal{B}_1(X)}$. In this case, we say that φ is a *one-block* flip and τ is the *symbol map* of φ. The following lemma states that every flip for a shift space can be recoded to a one-block flip.

Lemma 6.1.4. *Suppose that X is a shift space and φ is a flip for (X, σ_X). Then there are a finite set $\mathcal{A}$, a shift space Y over $\mathcal{A}$, and a one-block flip ψ for (Y, σ_Y) such that (Y, σ_Y, ψ) is conjugate to (X, σ_X, φ).*

Proof. Let $\mathcal{A} = \{(x_0, \varphi(x)_0) : x \in X\}$. Then $\mathcal{A}$ is finite since it is a subset of $\mathcal{B}_1(X) \times \mathcal{B}_1(X)$. We define two maps $\Phi : X \to \mathcal{A}^{\mathbb{Z}}$ and $\tau : \mathcal{A} \to \mathcal{A}$ by

$$\Phi(\langle x_i \rangle_{i \in \mathbb{Z}}) = \langle (x_i, \varphi(x)_{-i}) \rangle_{i \in \mathbb{Z}} \quad \text{and} \quad \tau(a, b) = (b, a).$$

If we set $Y = \{\Phi(x) : x \in X\}$, then it is clear that Y is a shift space over $\mathcal{A}$ and Φ is a conjugacy between X and Y. We define $\psi(y)_i = \tau(y_{-i})$ for $y \in Y$ and $i \in \mathbb{Z}$. Then since $\psi(\Phi(x)) = \Phi(\varphi(x))$, we conclude that ψ is a one-block flip for (Y, σ_Y) and Φ is a conjugacy between (X, σ_X, φ) and (Y, σ_Y, ψ). □

Suppose that φ is a one-block flip for (X, σ_X), and that τ is the symbol map. For notational simplicity, we write

$$\begin{aligned} \tau(a) &= a^* && (a \in \mathcal{B}_1(X)) \\ w^* &= w_n^* \cdots w_2^* w_1^* && (w = w_1 w_2 \cdots w_n \in \mathcal{B}_n(X)). \end{aligned}$$

Hence, for $x \in X$ and $i \leq j$, we have $\varphi(x)_{[i,j]} = (x_{[-j,-i]})^*$. It is obvious that $(w^*)^* = w$, and that $w^* \in \mathcal{B}(X)$ whenever $w \in \mathcal{B}(X)$, and that w^* is finitary whenenver w is finitary.

In addition, suppose that X has a finitary block f and $N = |f|$. We consider the Nth higher block system $(X^{[N]}, \sigma_{X^{[N]}})$ of (X, σ_X) and the Nth higher block code Ψ_N such that Ψ_N is a conjugacy between (X, σ_X) and $(X^{[N]}, \sigma_{X^{[N]}})$ (Remark 2.1.2(2)). It is clear that f is a finitary symbol of $X^{[N]}$. For this N we define $\varphi^{[N]} : X^{[N]} \to X^{[N]}$ by

$$\varphi^{[N]}(y)_i = (y_{-i})^*.$$

Then $\varphi^{[N]}$ is a one-block flip for $(X^{[N]}, \sigma_{X^{[N]}})$ and $\varphi^{[N]} \circ \Psi_N = \sigma_{X^{[N]}}^{-N+1} \circ \Psi_N \circ \varphi$. If N is odd, then $(X^{[N]}, \sigma_{X^{[N]}}, \varphi^{[N]})$ is conjugate to (X, σ_X, φ) under the conjugacy $\Psi_N \circ \sigma_X^{(1-N)/2}$, otherwise it is conjugate to (X, σ_X, φ) under the conjugacy $\Psi_N \circ \sigma_X^{-N/2}$. Thus we obtain the following lemma:

Lemma 6.1.5. *Suppose that (X, σ_X, φ) is a shift-flip system, and that X has a finitary block. There is a shift-flip system (Y, σ_Y, ψ) such that it conjugate to (X, σ_X, φ), Y has a finitary symbol and ψ is a one-block flip for (Y, σ_Y).*

In our proof of the propositions, the following lemma will play a crucial role.

Lemma 6.1.6. *Suppose that X is an irreducible shift space, $|X| = \infty$, and $f \in \mathcal{B}_1(X)$. Then there are blocks $a, b \in \mathcal{B}(X)$ such that*

(1) *$faf, fbf \in \mathcal{B}(X)$,*

(2) *f does not occur in a and b, and*

(3) *$fbfa$ does not occur in any $(|a|+1)$-periodic point.*

Proof. Since X is irreducible and $|X| = \infty$, there are blocks a and b such that faf, $fbf \in \mathcal{B}(X)$ and the following hold:

(i) if $fa'f \in \mathcal{B}(X)$, then $|a| \leq |a'|$,

(ii) $b \neq (af)^n a$ for all $n \geq 0$, and

(iii) if $fb'f \in \mathcal{B}(X)$ and $b' \neq (af)^n a$ for all $n \geq 0$, then $|b| \leq |b'|$.

It is then easy to see that the blocks a and b have the desired properties. □

The following lemma provides a sufficient condition for a flip φ to have the property that $A(\varphi) \neq \emptyset$.

Lemma 6.1.7. *Suppose that X is an infinite synchronized system and φ is a flip for (X, σ_X). If there is a point $x \in X$ such that $\varphi(x) = x$ and a finitary block occurs in x, then $A(\varphi) \neq \emptyset$.*

Proof. Suppose that $x \in X$, $\varphi(x) = x$ and that a finitary block occurs in x. By Lemma 6.1.4, we may assume that φ is a one-block flip for (X, σ_X). If n is sufficiently large, then $x_{[-n,n]}$ is finitary and $(x_{[-n,n]})^* = x_{[-n,n]}$ since a finitary block occurs in x and $\varphi(x) = x$. By Lemma 6.1.5 with $2n+1$, we may assume that there is a finitary symbol f with $f^* = f$ (let $f = x_{[-n,n]}$). Let $a, b \in \mathcal{B}(X)$ satisfy the conditions of Lemma 6.1.6. Choose a positive integer N such that

$$2|a| + 1 + 2(|b|+1) \leq (N-1)(|a|+1). \tag{6.1}$$

We put

$$w = afa(fb)^2(fa)^{2N} \quad \text{and} \quad M = (N+1)(|a|+1) + |b|,$$

then $|w| = 2M+1$. Since f is finitary and $f^* = f$, we have $(a^*f)^k w (fa)^k \in$

$\mathcal{B}(X)$ for all $k \geq 0$. Let

$$\begin{aligned} y_{(-\infty,-M-1]} &= \cdots a^*fa^*fa^*f \\ y_{[-M,M]} &= w = afa(fb)^2(fa)^{2N} \\ y_{[M+1,\infty)} &= fafafa\cdots, \end{aligned}$$

it is clear that $y \in X$, f occurs infinitely often in y, and $\varphi(y)_i = y_i$ if $|i| \geq M+1$. From 6.1

$$|afa(fb)^2| \leq |(a^*f)^{N-1}|$$

and since $fbfa$ can not occur in any $(|a|+1)$-periodic point, $w^* \neq w$. Hence $\varphi(y)_i \neq y_i$ for some $i \in [-M, M]$. The proof is done. □

Proof of Proposition 6.1.2. Suppose that X is an infinite synchronized system, φ is a flip for (X, σ_X) and f is a finitary block. By Lemma 6.1.4, we may assume that φ is a one-block flip. Then f^* is finitary, and there are blocks $w, u \in \mathcal{B}(X)$ with f^*wf, $fuf \in \mathcal{B}(X)$ since X is irreducible.

We first suppose that $|w| = 2N+1$. For any $k \geq 0$, $(u^*f^*)^k w(fu)^k \in \mathcal{B}(X)$, so there is $x \in X$ such that

$$\begin{aligned} x_{(-\infty,-N-1]} &= \cdots u^*f^*u^*f^* \\ x_{[-N,N]} &= w \\ x_{[N+1,\infty)} &= fufufu\cdots, \end{aligned}$$

Then it is obvious that a finitary block occurs infinitely often in x, $\varphi(x)_i = x_i$ if $|i| \geq N+1$. If $\varphi(x)_{[-N,N]} \neq w$, then $\varphi(x) \neq x$ and $x \in A(\varphi)$. If $\varphi(x)_{[-N,N]} \neq w$, then by Lemma 6.1.7 $A(\varphi)$ is not empty.

We now suppose that $|w| = 2N$. As the above there is a point $x \in X$ such that

$$\begin{aligned} x_{(-\infty,-N-1]} &= \cdots u^*f^*u^*f^* \\ x_{[-N,N-1]} &= w \\ x_{[N,\infty)} &= fufufu\cdots. \end{aligned}$$

Then $\sigma_X\varphi(x)_i = x_i$ for all $|i| \geq N+1$ and $i = N$. Also, $w^* = w$ if and only if $\sigma_X\varphi(x) = x$. As the above case $A(\sigma_X\varphi(x))$ is not empty. $\square$

From now on, for $A, B \subseteq \mathbb{Z}$ and $m \in \mathbb{Z}$ we write $mA = \{mj : j \in A\}$, $A + B = \{j + k : j \in A \text{ and } k \in B\}$, $(-1)A = -A$ and $\{m\} + A = m + A$.

Suppose that X is irreducible and has a finitary symbol f, and that φ is a one-block flip for (X, σ_X). Let $a, b \in \mathcal{B}(X)$ satisfy the conditions in Lemma 6.1.6. If there is a block $c \in \mathcal{B}(X)$ such that $|c| = 2\alpha + 1$, $f^*cf \in \mathcal{B}(X)$ and $c^* \neq c$, we choose a positive number N so that

$$|a| + 2|b| + |c| + 2 \leq (N-1)(|a|+1), \tag{6.2}$$

and put $d = fb(fa)^N$ and $\beta = \alpha + |d|$. Then

(I) d and d^* are finitary blocks,

(II) $d^*cd, d^*c^*d \in \mathcal{B}(X)$, $d^*cd \neq d^*c^*d$, and

(III) $|d^*cd| = |d^*c^*d| = 2\beta + 1$,

since f, f^* are finitary blocks and $c \neq c^*$. From Lemma 6.1.6(2) and (3) we obtain the following lemma:

Lemma 6.1.8. *Let $x \in X$ and $i \neq j$. If $x_{[i-\beta, i+\beta]}, x_{[j-\beta, j+\beta]} \in \{d^*cd, d^*c^*d\}$, then $|i - j| \geq |c| + |d| + 1$.*

For $x \in X$ let $\mathcal{M}(x)$ denote the set of integers i such that $x_{[i-\beta,i+\beta]} \in \{d^*cd, d^*c^*d\}$. From Lemma 6.1.8 we have

$$[i - \alpha - 1, i + \alpha + 1] \cap [j - \beta, j + \beta] = \emptyset \qquad (i, j \in \mathcal{M}(x),\ i \neq j) \tag{6.3}$$

for every $x \in X$. In particular, the intervals $[i - \alpha, i + \alpha]$, $i \in \mathcal{M}(x)$, are mutually disjoint. Therefore, for each $i \in \mathcal{M}(x) + [-\alpha, \alpha]$, there is the unique number $c(i; x)$ in $\mathcal{M}(x)$ such that $i \in [c(i; x) - \alpha, c(i; x) + \alpha]$. Note that $x_{[c(i;x)-\alpha, c(i;x)+\alpha]} = c$ or c^*.

Let $A \subset \mathbb{Z}$ and $x \in X$. We define the bi-infinite sequence $\theta_A(x)$ by

$$\theta_A(x)_i = \begin{cases} (x_{2c(i;x)-i})^* & \text{if } i \in (\mathcal{M}(x) \cap A) + [-\alpha, \alpha] \\ x_i & \text{otherwise.} \end{cases} \tag{6.4}$$

If $j \in \mathcal{M}(x) \cap A$, then $c(j+k; x) = j$ for $k \in [-\alpha, \alpha]$ and $\theta_A(x)_{j+k} = (x_{j-k})^*$; hence $\theta_A(x)_{[j-\alpha,j+\alpha]} = (x_{[j-\alpha,j+\alpha]})^*$. Thus θ_A replaces the part $x_{[i-\alpha,i+\alpha]}$ of x with $(x_{[i-\alpha,i+\alpha]})^*$ whenever $i \in \mathcal{M}(x) \cap A$ and leaves the remaining part of x unchanged. Since d and d^* are finitary blocks and $d^*cd, d^*c^*d \in \mathcal{B}(X)$, we have $\theta_A(x) \in X$ for all $x \in X$. From this and (6.3), it follows that $\mathcal{M}(x) = \mathcal{M}(\theta_A(x))$, and consequently $\theta_A(\theta_A(x)) = x$ for all $x \in X$. If $x, x' \in X$, $i \in \mathbb{Z}$ and $x_{[i-\alpha-\beta,i+\alpha+\beta]} = x'_{[i-\alpha-\beta,i+\alpha+\beta]}$, then $\theta_A(x)_i = \theta_A(x')_i$. Hence $\theta_A : X \to X$ is a homeomorphism satisfying $\theta_A^2 = \mathrm{id}_X$ for every $A \subset \mathbb{Z}$.

Lemma 6.1.9. *Let $A, B \subseteq \mathbb{Z}$. Then*

(1) $\theta_A\theta_B = \theta_{(A\triangle B)} = \theta_B\theta_A$,

(2) $\sigma_X\theta_A = \theta_{(-1+A)}\sigma_X$, *and*

(3) $\varphi\theta_A = \theta_{(-A)}\varphi$.

Proof. If $i \in \mathcal{M}(x) \cap (A \cap B)$, then $\theta_A(\theta_B(x))_{[i-\alpha,i+\alpha]} = x_{[i-\alpha,i+\alpha]}$. Hence $\theta_A\theta_B = \theta_{(A\triangle B)}$, and (1) hold. Since $\mathcal{M}(x) = \mathcal{M}(\sigma(x))$, $i \in \mathcal{M}(x) \cap A$ if and only if $i - 1 \in \mathcal{M}(\sigma_X(x)) \cap (-1 + A)$. If $i \in \mathcal{M}(x) \cap A$, then for $k \in [-\alpha, \alpha]$,

$$\sigma_X\theta_A(x)_{i+k-1} = (x_{i-k})^* = (\sigma_X(x)_{i-1-k})^* = \theta_{(-1+A)}\sigma_X(x)_{i+k-1},$$

hence the statement (2) hold. The statement (3) hold since $\mathcal{M}(\varphi(x)) = -\mathcal{M}(x)$. $\square$

Proof of Proposition 6.1.3. Suppose that X is an infinite synchronized system, φ is a flip for (X, σ_X) and $A(\varphi) \neq \emptyset$. We will construct an automorphism θ of (X, σ_X) and homeomorphisms $\theta_1, \theta_2, \theta_3, \ldots$ from X onto itself such that $\theta\varphi$ is a flip for (X, σ_X), and the following hold:

(i) $\theta_n(F(\varphi; n)) \subseteq F(\theta\varphi; n)$ for all n,

(ii) $\theta_n(F(\varphi;n)) \neq F(\theta\varphi;n)$ for some n, and

(iii) $A(\theta\varphi) \neq \emptyset$

From Lemma 6.1.5 we may assume that φ is a one-block map and f is a finitary symbol. Let $a, b \in \mathcal{B}(X)$ satisfy the conditions of Lemma 6.1.6. Since X is irreducible and $A(\varphi) \neq \emptyset$, there is a block c of length $2|\alpha|+1$ such that $f^*cf \in \mathcal{B}(X)$ and $c^* \neq c$. We choose a positive number N and a block d as the above argument. Here we use the same notaitions in the above argument. For $n = 1, 2, 3, \cdots$ we set

$$H(n) = \bigcup_{k\in\mathbb{Z}} \left\{ i \in \mathbb{Z} : nk < i < n\left(k + \frac{1}{2}\right)\right\}.$$

From (6.4), we define $\theta = \theta_Z$ and $\theta_n = \theta_{H(n)}$ for $n = 1, 2, 3, \cdots$. It is clear that θ is an automorphism of (X, σ_X) such that $\theta^2 = \mathrm{id}_X$ and $\theta\varphi = \varphi\theta$ from Lemma6.1.9(2) and (3). Thus $\theta\varphi$ is also a flip for (X, σ_X).

To prove (i), suppose that n is a positive integer and $x \in F(\varphi;n)$, that is, $\sigma_X^n(x) = \varphi(x) = x$. By definition of $H(n)$ we have $-n + H(n) = H(n)$, and by Lemma 6.1.9(2) we have $\sigma_X^n \theta_{H(n)} = \theta_{(-n+H(n))}\sigma_X^n$. Hence

$$\sigma_X^n(\theta_n(x)) = \theta_n(x). \tag{6.5}$$

If we put $C = \mathbb{Z} \setminus (H(n) \cup (-H(n)))$, then $\{H(n), -H(n), C\}$ is a partition of $\mathbb{Z}$. Since $C = \{nk, nk + n/2 : k \in \mathbb{Z}\} + \mathbb{Z}$, we have $C = n\mathbb{Z}$ in the case when n is odd, and $C = n\mathbb{Z} + \{0, n/2\}$ in the case when n is even. If $\mathcal{M}(x) \cap C$ is not empty, then $0 \in \mathcal{M}(x) \cap C$ or $n/2 \in \mathcal{M}(x) \cap C$ since $\sigma_X^n(x) = x$. Since $\varphi(x) = x$, $x_0 = (x_0)^*$ or $x_{n/2} = (x_{n/2})^*$. However these are impossible since $c^* \neq c$, hence $\mathcal{M}(x) \cap C = \emptyset$. Thus we have

$$\mathcal{M}(x) \cap (\mathbb{Z} \triangle (-H(n))) = \mathcal{M}(x) \cap (H(n) \cup C) = \mathcal{M}(x) \cap H(n),$$

so that

$$\theta\varphi(\theta_n(x)) = \theta\theta_{(-H(n))}\varphi(x) = \theta_{(\mathbb{Z}\triangle(-H(n)))}(x) = \theta_n(x).$$

From this and (6.5) we obtain $\theta_n(x) \in F(\theta\varphi;n)$ and (i) is proved.

To prove (ii) and (iii), we first construct a point in X. Since X is irreducible, there is a block w such that $dwd^* \in \mathcal{B}(X)$. If we write $2(|c|+2|d|+|w|) = n$ and $|c|+2|d|+n = 2m+1$, then we have $n = |w^*d^*cdwd^*cd|$, $w^*d^*cdwd^*cd \in \mathcal{B}_n(X)$, and there is a point $z \in X$ such that $\sigma_X^n(z) = z$ and

$$z_{[-m,m]} = d^*cdw^*d^*cdwd^*cd$$

as d and d^* are finitary blocks. Then $\varphi(z) \neq z$ because $z_{[-\alpha,\alpha]} = c$. Hence $z \notin F(\varphi; n)$. Since $\sigma_X^n(z) = z$ and $\sigma_X^n \theta_{H(n)} = \theta_{(-n+H(n))}\sigma_X^n$, we have $\sigma_X^n(\theta_n(z)) = \theta_n(z)$. Let C be as in the proof of (i). Then $C = n\mathbb{Z} + \{0, n/2\}$ and $C \subseteq \mathcal{M}(z)$ and we have $\varphi(z) = \theta_C(z)$, hence

$$\theta\varphi(\theta_n(z)) = \theta\theta_{(-H(n))}\varphi(z) = \theta\theta_{(-H(n))}\theta_C(z) = \theta_n(z).$$

Thus $\theta_n(z) \in F(\theta\varphi; n)$, while $z \notin F(\varphi; n)$. Since θ_n is one-to-one, we see that (ii) holds. Finally, we have $\theta\varphi(\theta_n(z)) = \theta_n(z)$, and the finitary block d occurs in $\theta_n(z)$, hence (iii) follows from Lemma 6.1.7. This proves the proposition. □

6.2 Flips for a certain coded system

We recall that a coded system X is a shift space which has a code $\mathcal{C}$ such that the set of bi-infinite concatenations of blocks from $\mathcal{C}$ is dense in X.

Recall that the mirror map ρ is a flip for (X, σ_X) if X is closed under ρ. This section is devoted to prove the following:

Theorem 6.2.1. *There is an infintie coded system X such that the mirror map ρ is a flip for (X, σ_X) and $\{\sigma_X^m\rho : m \in \mathbb{Z}\}$ is the set of flips for (X, σ_X).*

Corollary 6.2.2 is an immediate consequence of Theorem 6.2.1.

Corollary 6.2.2. *X has only two non-conjugate flips $\sigma_X\rho$ and ρ.*

To construct X in Theorem 6.2.1 we follow the method given in [FieF2, Section 1]. In [FieF2], they define a set $\mathcal{C}$ of blocks which are stable, neutral, and have a proper skeleton. This set $\mathcal{C}$ induces a coded system whose automorphism group is generated by the shift map and is isomorphic to $\mathbb{Z}$ [FieF2, Corollary 2.2]. Fortunately we can simplify the construction: we use only the stability of blocks to define a code. We introduce the definition of the stability of blocks (See [FieF2] for definitions of neutrality and proper skeleton). Let $\mathcal{A} = \{0, 1, 2\}$,

$$\begin{aligned} I &= \bigcup_{k \geq 1} [2^{2k}, 2^{2k+1}] = [4, 8] \cup [16, 32] \cup \cdots, \text{and} \\ J &= \{(0,0), (1,1), (1,2), (2,1), (2,2)\}. \end{aligned}$$

Definition 6.2.3. [FieF2] A block $w \in \mathcal{B}(\mathcal{A}^{\mathbb{Z}})$ is stable if it satisfies the following conditions:

(1) the blocks 12 and 21 do not occur in w,

(2) if $x \in \mathcal{A}^{\mathbb{Z}}$, $x_{[1,3|w|+2]} = 0^{|w|+1} w 0^{|w|+1}$, $1 \leq n \leq |w|$, $a \in \{1, 2\}$, $|w| + 1 \leq i < j \leq 2|w| + 2$, and $x_{[i,j]} = 0a0$, then

$$a = 1 \text{ if and only if } (n, (x_{i-n}, x_{j+n})) \in (I \times J) \cup (I^C \times J^C).$$

In the above condition (2) it does not matter of the position of $0^{|w|+1} w 0^{|w|+1}$ in x since $\mathcal{A}^{\mathbb{Z}}$ is shift-invariant. Here are some easy consequences of the definitions. We present the proof here for the reader's convenience.

Lemma 6.2.4.

(1) *Let $j \neq 0$ be an integer. The sets $\{n : n, n+j \in I\}$, $\{n : n \in I, n+j \notin I\}$, $\{n : n \notin I, n+j \in I\}$ and $\{n : n, n+j \notin I\}$ are all infinite.*

(2) *0^n is stable for all n, 1^n is stable if and only if $n \in I$, and 2^n is stable if and only if $n \in I$.*

(3) *If $w = w_1 w_2 \cdots w_{|w|}$ is stable, then so is the reversed block $w_{|w|} \cdots w_2 w_1$.*

(4) *If w is stable, then $0w$ and $w0$ are stable. Furthermore $0^{|w|+1} w 0^{|w|+1}$ is stable.*

(5) *If w, w' are stable and $n \geq \max\{|w|, |w'|\}$, then $w0^n w'$ is stable.*

Proof. (1) Let $j \neq 0$ be an integer. We will only show that $\{n : n, n+j \in I\}$ is infinite since the others can be proved similarly. If we write $I = \bigcup_{k\geq 1} I_k$ where $I_k = [2^{2k}, 2^{2k+1}]$, and put $|I_k| := 2^{2k+1} - 2^{2k} = 2^{2k}$, then $|I_1| < |I_2| < \cdots$. Choose K with $|j| < 2^{2K}$. Let $k \geq K+1$. If $j > 0$, for $n \in [2^{2k}, 2^{2k}+j]$, n, $n+j \in I_k$ since $|I_k| > |I_K|$. Similarly if $j < 0$, then $n, n+j \in I_k$ for $n \in [2^{2k+1}+j, 2^{2k+1}]$. Hence for each $k > K$ there is n such that n, $n+j \in I_k$, so that we obtain the desired result.

(2) It is evident that 0^n is stable for all n. Let $a \in \{1,2\}$, $n = 1, 2, \cdots$, and $x \in \mathcal{A}^{\mathbb{Z}}$ with $x_{[1,3n+2]} = 0a^n0$. Then $x_{[n+1,2n+2]} = 0a^n0$ and $(x_{n+1-n}, x_{2n+2+n}) = (x_1, x_{3n+2}) = (0,0) \in J$. Thus 1^n is stable if and only if $n \in I$, and 2^n is stable if and only if $n \notin I$.

(3) Suppose that $w = w_1 w_2 \cdots w_n$ is stable. Let $w' = w_n \cdots w_2 w_1$. It is evident that w' does not contain the blocks 12 and 21. Let $x \in \mathcal{A}^{\mathbb{Z}}$ with $x_{[1,3n+2]} = 0^{n+1}w'0^{n+1}$. Since the mirror map ρ is a flip for $(\mathcal{A}^{\mathbb{Z}}, \sigma_{\mathcal{A}})$, $y = \rho(x) \in \mathcal{A}^{\mathbb{Z}}$ and $y_{[-3n-2,-1]} = 0^{n+1}w0^{n+1}$. If $x_{[i,j]} = 0a^m0$ for $a \neq 0$, $1 \leq m \leq n$, $n+1 \leq i < j \leq 2n+2$, then $y_{[-j,-i]} = 0a^m0$. Since w is stable and $(e,f) \in J \Leftrightarrow (f,e) \in J$, $a = 1$ if and only if $(m, (y_{s-m}, y_{t+m})) \in (I \times J) \cup (I^C \times J^C)$ if and only if $(m, (x_{i-m}, x_{j+m})) \in (I \times J) \cup (I^C \times J^C)$. Thus w' is stable.

(4) Let w be stable. It suffices to show that $0w$ satisfies the condition (2) of Definition 6.2.3. If $x \in \mathcal{A}^{\mathbb{Z}}$ and $x_{[1,3|0w|+2]} = 0^{|0w|+1}0w0^{|0w|+1}$, then $x_{[3,3|w|+4]} = 0^{|w|+1}w0^{|w|+1}$. If $0a^m0$ occurs in $x_{[1,3|0w|+2]}$, then it must occur in $x_{[3,3|w|+4]}$. Since w is stable, the condition (2) of Definition 6.2.3 hold.

(5) Suppose that w, w' are stable, and that $n \geq \max\{|w|, |w'|\}$. Let $u = w0^{n+1}w'$. It is evident that 12 and 21 do not occurs in u. Let $x \in \mathcal{A}^{\mathbb{Z}}$ with $x_{[1,3|u|+2]} = 0^{|u|+1}u0^{|u|+1}$. If $a \neq 0$ and $0a^m0$ occurs in $x_{[1,3|u|+2]}$ for some m, then it occurs in either $0^{|w|+1}w0^{|w|+1}$ or $0^{|w'|+1}w'0^{|w'|+1}$. Since w, w' are stable

and $n > |w|, |w'|$, the condition (2) of Definition 6.2.3 hold. $\square$

We set
$$\mathcal{C} = \{0\} \cup \{0^{|w|+1} w 0^{|w|+1} : \ w \text{ is stable}\},$$
and define $W(\mathcal{C})$ to be the set of bi-infinite concatenations of blocks from $\mathcal{C}$. We define X to be the coded system for which $\mathcal{C}$ is a code, i.e., $W(\mathcal{C})$ is a dense subset of X.

Lemma 6.2.5. *Every finite concatenation w of blocks from $\mathcal{C}$ is stable. Hence w is a subblock of a block in $\mathcal{C}$.*

Proof. Let w be a finite concatenation of blocks from $\mathcal{C}$. If $w = 0^{|w|}$, then it is stable by Lemma 6.2.4(2). Suppose that

$$\begin{aligned} w &= 0^{m_0}\overline{u_1}0^{m_1}\overline{u_2}\cdots 0^{m_{k-1}}\overline{u_k}0^{m_k} \\ &= 0^{m_0+n_1+1}u_1 0^{n_1+m_1+n_2+2}u_2\cdots 0^{n_{k-1}+m_{k-1}+n_k+2}u_k 0^{m_k+n_k+1} \end{aligned}$$

where u_j is stable, $|u_j| = n_j$, and $\overline{u_j} = 0^{n_j+1}u_j 0^{n_j+1}$. The length of the concatenation of 0's between u_j and u_{j+1} is larger than $|u_j|, |u_{j+1}|$. The same argument in the proof of Lemma 6.2.4(5) shows that

$$u_1 0^{n_1+m_1+n_2+2}u_2\cdots 0^{n_{k-1}+m_{k-1}+n_k+2}u_k =: v$$

is stable. If we apply Lemma 6.2.4(5) to v repeatedly, then w is stable and $0^{|w|+1}w0^{|w|+1} \in \mathcal{C}$. $\square$

From the definition of X and $W(\mathcal{C})$ we obtain the following:

Lemma 6.2.6. *For $x \in \mathcal{A}^{\mathbb{Z}}$ the following are equivalent.*

(1) *$x \in X$.*

(2) *For all n there is $y \in W(\mathcal{C})$ such that $y_{[-n,n]} = x_{[-n,n]}$.*

(3) *For all n there is a finite concatenation w of blocks from $\mathcal{C}$ such that $x_{[-n,n]}$ is a subblock of w.*

(4) *For all* n *there is a stable block* w *such that* $x_{[-n,n]}$ *is a subblock of* w.

Proof. Since $W(\mathcal{C})$ is dense in X, (1) and (2) are equivalent. We will prove that (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (2). Let $n \in \mathbb{N}$ and $y \in W(\mathcal{C})$ with $x_{[-n,n]} = y_{[-n,n]}$. Since $y_{[-n,n]}$ is a subblock of a finite concatenation of blocks from $\mathcal{C}$, (3) hold. From Lemma 6.2.5, (3) implies that (4). Finally suppose that $n \in \mathbb{N}$, and that $x_{[-n,n]}$ is a subblock of a stable block w. Since $0^\infty w 0^\infty \in W(\mathcal{C})$, the statement (2) hold. □

Lemma 6.2.6 guarantees all points in X satisfy Definition 6.2.3.

Lemma 6.2.7. *Let* $x \in X$. *Then the blocks* 12 *and* 21 *do not occur in* x. *If* $i < j$, $a \in \{1, 2\}$, $n > 0$, *and* $x_{[i,j]} = 0a^n0$, *then* $a = 1$ *if and only if* $(n, (x_{i-n}, x_{j+n})) \in (I \times J) \cup (I^C \times J^C)$.

Proof. Let $x \in X$. From Lemma 6.2.6(4), the blocks 12 and 21 do not occur in x. Suppose that $i < j$, $a \in \{1, 2\}$, $n > 0$, and $x_{[i,j]} = 0a^n0$. Choose a positive number N so that $x_{[i-n,j+n]}$ is a subblock of $x_{[-N,N]}$.

By Lemma 6.2.6(4), there are a stable block w and a point $y \in \mathcal{A}^{\mathbb{Z}}$ such that

(i) $y_{[1,3|w|+2]} = w$,

(ii) $x_{[-N,N]}$ is a subblock of w,

(iii) $y_{[s-n,t+n]} = x_{[i-n,j+n]}$.

Then $a = 1 \Leftrightarrow (n, (y_{s-n}, y_{t+n})) \in (I \times J) \cup (I^C \times J^C) \Leftrightarrow (n, (x_{i-n}, x_{j+n})) \in (I \times J) \cup (I^C \times J^C)$. □

Let X_0 denote the set of $x \in X$ such that $|\{i \in \mathbb{Z} : x_i \neq 0\}| < \infty$.

Lemma 6.2.8. *The set* X_0 *is a dense subset of* X.

Proof. Let $x \in X$ and $n > 0$. From Lemma 6.2.6(4) there is a stable block w such that $x_{[-n,n]}$ is a subblock of w. If we put $y = 0^\infty w 0^\infty$ and $y_{[-n,n]} = x_{[-n,n]}$, then $y \in X_0$. Since n is arbitrary, X_0 is dense in X. □

For $x \in X$, we put $\mathcal{Z}(x) = \{i \in \mathbb{Z} : x_i = 0\}$. If $x \in X_0$, then $\mathcal{Z}(x)$ determines x.

Proposition 6.2.9. *If $x, x' \in X_0$ and $\mathcal{Z}(x) = \mathcal{Z}(x')$, then $x = x'$.*

Proof. Suppose that $x, x' \in X_0$ and $\mathcal{Z}(x) = \mathcal{Z}(x')$. To obtain a contradiction, suppose that $x_i \neq x'_i$. We may assume that $i = 0$ since $\mathcal{Z}(\sigma_X^i x) = \mathcal{Z}(\sigma_X^i x')$. We may also assume that $x_0 = 1$ and $x'_0 = 2$. Since $\mathcal{Z}(x) = \mathcal{Z}(x')$, the first assertion of Lemma 6.2.7 implies that

$$x_{[-i,j]} = 01^n0 \quad \text{and} \quad x_{[-i,j]} = 02^n0.$$

for some $-i, j \in \mathbb{Z}$. Since $\mathcal{Z}(x) = \mathcal{Z}(x')$,

$$(x_{-i-n}, x_{j+n}) \in J \Leftrightarrow (y_{-i-n}, y_{j+n}) \in J.$$

By the second assertion of Lemma 6.2.7, $1 = x_0 = y_0 = 2$, it is a contradiction. □

The following proposition states that the automorphism group of X is generated by the shift map σ_X. This is a simplified version of Proposition 1.6 in [FieF2]. Although its proof is essentially same as the one of Proposition 1.6 in [FieF2], we present the proof here for the reader's convenience.

Proposition 6.2.10. *The automorphism group of (X, σ_X) is $\{\sigma_X^m : m \in \mathbb{Z}\}$.*

Proof. We will show that for each automorphism θ of (X, σ_X) there is an integer m such that

$$\mathcal{Z}(\sigma_X^m \theta(x)) = \mathcal{Z}(x) \qquad (x \in X_0). \tag{6.6}$$

Suppose that θ is an automorphism of (X, σ_X). There are a non-negative integer N and a block map $\Theta : \mathcal{B}_{2N+1}(X) \to \mathcal{B}_1(X)$ such that

$$\theta(x)_i = \Theta(x_{[i-N,i+N]}) \qquad (x \in X, i \in \mathbb{Z}). \tag{6.7}$$

It is obvious that $0^\infty, 1^\infty, 2^\infty \in X$ and $\theta(\{0^\infty, 1^\infty, 2^\infty\}) = \{0^\infty, 1^\infty, 2^\infty\}$.

First, we show that $\theta(0^\infty) = 0^\infty$. To obtain a contradiction suppose that $\theta(0^\infty) \neq 0^\infty$. There are $a, b \in \{1, 2\}$ such that $\theta(a^\infty) = 0^\infty$ and $\theta(0^\infty) = b^\infty$. Choos a positive number M so that a^M is stable and $M \geq 2N + 1$. Let $n \geq M + 1$. Then $a^M 0^n a^M$ is stable(Lemma 6.2.4(5)) and

$$x := 0^\infty.a^M 0^n a^M 0^\infty \in W(\mathcal{C}) \subseteq X.$$

Since $\theta(a^\infty) = 0^\infty$, $\theta(0^\infty) = b^\infty$ and $M \geq 2N+1$, there are blocks u_1, u_2, u_3, u_4 such that

$$\theta(x) = b^\infty u_1 0^{M-2N} u_2 b^{n-2N} u_3 0^{M-2N} u_4 b^\infty \tag{6.8}$$

from (6.7). Since $12, 21$ do not occur in any point of X(Lemma 6.2.7) and $\theta(x) \in X$, we have

$$u_1 0^{M-2N} u_2 = u0b^{k_1} \qquad \text{and} \qquad u_3 0^{M-2N} u_4 = b^{k_2} 0v \tag{6.9}$$

for some blocks u, v, $0 \leq k_1 \leq |C|$ and $0 \leq k_2 \leq |D|$. Combining (6.8) and (6.9), there is an integer k such that

$$\sigma_X^k \theta(x) = b^\infty u.0b^{n+j}0vb^\infty$$

where $j = -2N + k_1 + k_2$. This equation hold for all $n \geq M + 1$, and the value j is fixed for any $n \geq M + 1$. Let $y = \theta(x)$, then $y_{[0,n+j+1]} = 0b^{n+j}0$. If $n + j > \max\{|u|, |v|\}$, then $y_{[-n-j,2(n+j)+1]} = (b, b) \in J$. Hence, by Lemma 6.2.7, if $n+j \in I$ then $b = 1$. Otherwise $b = 2$. However $n+j \in J$ for infinitely many n, and also $n + j \in J$ for infinitely many n(Lemma 6.2.4(1)). It is a contradiction. Thus $\theta(0^\infty) = 0^\infty$.

Since $\theta(0^\infty) = 0^\infty$, we have $\theta(1^\infty) = c^\infty$ for some $a \in \{1, 2\}$. If $n \in I$, then 1^n is stable(Lemma 6.2.4(2)), and so that $0^\infty.01^n0^\infty \in X$. As the above argument there are blocks u, v and integers l, m such that $l \geq -2N$ and

$$\sigma_X^m \theta(0^\infty.01^n 0^\infty) = 0^\infty p.0c^{n+l}0q0^\infty$$

for all $n \in I$ with $n \geq 2N + 1$. From Lemma 6.2.4(1), there are infinitely many $n \in I$ with $n + l \in I$. If $l \neq 0$, there are also infinitely many $n \in I$ with

$n+l \notin I$(Lemma 6.2.4(1)). By the same reason as in the proof of $\theta(0^\infty) = 0^\infty$, we conclude that $l = 0$ and $c = 1$. Thus, for $n \in I$ with $n \geq 2N + 1$

$$\sigma_X^m \theta(0^\infty.01^n 0^\infty) = 0^\infty p.01^n 0q0^\infty. \tag{6.10}$$

Now we will show that (6.6). Let $x \in X_0$. Suppose that $x_0 = 0$. We can choose a positive number L so that $L \geq N + |m|$ and $x_i = 0$ for $|i| > L$. That is, $x = 0^\infty x_{[-L,-1]}.0x_{[1,L]}0^\infty$, so that $x_{[-L,-1]}0x_{[1,L]} = x_{[-L,L]}$ is stable(Lemma 6.2.6). Let $n \in I$ and $n \geq \max\{3L+2, |q|+1\}$. Then $x_{[-L,L]}0^{n-L}1^n$ is stable. Indeed, let $y = 0^\infty x_{[-L,L]}0^{n-L}1^n 0^\infty$ with $y_{[-L,L]} = x_{[-L,L]}$. Since $n - L \geq 2L+2$ and 1^n is stable, we only consider $01^n 0 = y_{[n,2n+1]}$, and show that $y_{[0,3n+1]} \in J$. Since $y_{[0,3n+1]} = (x_0, 0) = (0,0) \in J$, thus $x_{[-L,L]}0^{n-L}1^n$ is stable.

Let

$$z = 0^\infty x_{[-L,L]}0^{n-L}1^n 0^\infty \quad \text{and} \quad z_{[-L.L]} = x_{[-L.L]}.$$

Then $z \in X_0$. Also, $\sigma_X^m \theta(x)_0 = \sigma_X^m \theta(z)_0$ since $z_{[-L.L]} = x_{[-L.L]}$ and $L \geq N + |m|$. From the assumptions of n and L, (6.10) implies that

$$\sigma_X^m \theta(z)_{[n,3n+1]} = 01^n 0q0^{n-|q|}.$$

Then $\sigma_X^m \theta(z)_{[n,2n+1]} = 01^n 0$, $\sigma_X^m \theta(z)_{3n+1} = 0$ and $n \in I$. From Lemma 6.2.7, $\sigma_X^m \theta(z)_0$ must be 0, so that $\sigma_X^m \theta(x)_0 = 0$. Since X_0 is shift-invariant, we obtain

$$\mathcal{Z}(x) \subseteq \mathcal{Z}(\sigma_X^m \theta(x)) \qquad (x \in X_0).$$

Similarly, for θ^{-1}, there is an integer m' such thtat

$$\mathcal{Z}(x) \subseteq \mathcal{Z}(\sigma_X^{m'} \theta^{-1}(x)) \qquad (x \in X_0).$$

We then have

$$\mathcal{Z}(x) \subseteq \mathcal{Z}(\sigma_X^m \theta(x)) \subseteq \mathcal{Z}(\sigma_X^{m'} \theta^{-1} \sigma_X^m \theta(x)) = \mathcal{Z}(\sigma_X^{m+m'}(x))$$

for $x \in X_0$. If $m + m' \neq 0$, then for

$$y = 0^\infty 1^{|m+m'|+1}.1^{n-|m+m'|-1}0^\infty \in X_0$$

with $n \in I$ and $n \geq |m| + |m'| + 1$, there is an integer i such that $i \in \mathcal{Z}(x)$ but $i \notin \mathcal{Z}(\sigma_X^{m+m'}(x))$. Thus $m + m' = 0$, and we obtain (6.6). Since $\theta(0^\infty) = 0^\infty$, $\sigma_X(X_0) = X_0$ and (6.6) hold, Proposition 6.2.9 implies that $\theta = \sigma_X^{-m}$ on X_0. From Lemma 6.2.8, $\theta = \sigma_X^{-m}$ on X_0. The proof is done. □

We now finish the proof of Theorem 6.2.1.

Proof of Theorem 6.2.1. It is obvious that $|X| = \infty$. Suppose that $x \in X$ and $\rho(x) = y$. Let $n \geq 0$. From Lemma 6.2.6 there is a stable block w such that $x_{[-n,n]}$ occurs in w. Then $y_{[-n,n]}$ occurs in the reversed block w' of w. By Lemma 6.2.4(3), the block w' is stable. Since n is arbitrary, $y \in X$ from Lemma 6.2.6. Thus X is closed under the mirror map ρ. It is clear that whenever φ is a filp for (X, σ_X) then $\varphi\rho$ is an automorphism of (X, σ_X). Proposition 6.2.10 implies that $\varphi = \sigma_X^m \rho$ for some m. □

6.3 Flips for the Morse shift

In this section we will show that the Morse shift has finitely manny non-conjugate flips.

For $a \in \{0, 1\}$, we put $\bar{a} = 1 - a$, and if $w = w_1 w_2 \cdots w_{|w|} \in \{0, 1\}^{|w|}$, we denote $\overline{w_1}\,\overline{w_2} \cdots \overline{w_{|w|}}$ by $\overline{w}$. The Morse shift is defined as follows [GotH, LinM, MorH]. We define inductively blocks $B_0, B_1, B_2, \cdots$ over $\{0, 1\}$ by $B_0 = 0$ and $B_n = B_{n-1}\overline{B_{n-1}}$ for $n = 1, 2, \cdots$. It is clear that B_n is a prefix of B_{n+1} and $|B_n| = 2^n$ for $n = 0, 1, 2 \cdots$. The *Morse sequence* $\mu \in \{0, 1\}^{\mathbb{Z}}$ is defined by

$$\mu_{[0,2^n-1]} = B_n \quad (n = 0, 1, 2, \cdots),$$
$$\mu_{-i} = \mu_{i-1} \quad (i = 1, 2, 3, \cdots).$$

Since B_n is a prefix of $B_{n+1}(n \geq 0)$, the right-infinite sequence $\mu_{[0,\infty)}$ is well-defined and $\mu \in \{0, 1\}^{\mathbb{Z}}$. A subshift (M, σ_M) of $(\{0, 1\}^{\mathbb{Z}}, \sigma)$ is the *Morse shift* if M is the closure of the set of $\{\sigma^m(\mu) : m \in \mathbb{Z}\}$. We set

$$\mathcal{F} = \{awawa : a \in \{0,1\} \text{ and } w \in \mathcal{B}(\{0,1\}^{\mathbb{Z}})\}.$$

Then $\mathcal{F}$ is a forbidden set of the Morse shift M, that is, $M = \mathbf{X}_{\mathcal{F}}$ [GotH, MorH]. In particular, if $x \in \mathbf{X}_{\mathcal{F}}$ then every subblock of x occurs in μ [GotH], and any block in $\mathcal{F}$ can not occur in μ [MorH].

The mirror map ρ is a flip for (M, σ_M) since $\rho(\mathbf{X}_{\mathcal{F}}) = \mathbf{X}_{\mathcal{F}}$. The map $\psi : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ defined by $\psi(x)_i = \overline{x_{-i}}$ is a flip for $(\{0,1\}^{\mathbb{Z}}, \sigma)$. Then ψ is also a flip for (M, σ_M) since $\psi(\mathbf{X}_{\mathcal{F}}) = \mathbf{X}_{\mathcal{F}}$.

We have $\rho\psi = \psi\rho$, and $\rho\psi =: \overline{\psi}$ is an automorphism of (M, σ_M) satisfying $\overline{\psi}^2 = \mathrm{id}_M$. In fact, $\overline{\psi}(x)_i = \overline{x_i}$ for $x \in M$ and $i \in \mathbb{Z}$. The goal of this section is the following:

Theorem 6.3.1. *Every flip for (M, σ_M) is conjugate to one of the four flips ρ, ψ, $\rho\sigma_M$ and $\psi\sigma_M$.*

This theorem is an immediate consequence of the following proposition:

Proposition 6.3.2. *The automorphism group of the Morse shift (M, σ_M) is generated by $\{\sigma_M, \overline{\psi}\}$.*

If φ is a flip for (M, σ_M), then Proposition 6.3.2 implies that $\rho\varphi$ is one of σ_M^{2m}, σ_M^{2m+1}, $\sigma_M^{2m}\overline{\psi}$ and $\sigma_M^{2m}\overline{\psi}$. If $\rho\varphi = \sigma_M^{2m}$, then $\sigma_M^m\varphi = \rho\sigma_M^m$ and φ is conjugate to ρ. The other cases are proved by similar method.

In the rest we shall prove Proposition 6.3.2. This proposition is shown in [Cov]. In fact, the Morse shift is also a substitution minimal set, and [Cov] provides the automorphism group of a substitution minimal set (We refer to [LinM] for the definition of substitution minimal sets). Nevertheless we prove Proposition 6.3.2 here; the method of proof is different from the one in [Cov]. We will use a result in [GotH]. Now we state and prove lemmas.

Lemma 6.3.3. [GotH, Lemma 4] Let $x \in X$ and let n be a non-negative integer. There is an integer k such that

$$x_{[k+m2^n,\, k+m2^n+2^n-1]} \in \{B_n, \overline{B_n}\} \quad (m \in \mathbb{Z}).$$

Proof. See [GotH]. $\square$

Remark 6.3.4. This lemma states that every point $x \in M$ can be represented by a bi-infinite concatenation of B_n and $\overline{B_n}$ for a non-negative integer n.

Lemma 6.3.5. *Let $x, y \in M$ and $N \geq 0$. There is an integer j such that $x_{[-N,N]} = y_{[j-N,j+N]}$.*

Proof. Let $x, y \in M$ and $N \geq 0$. There is an integer $n \geq 0$ such that $x_{[-N,N]}$ is a subblock of B_n. If we apply Lemma 6.3.5 to y and n, then y can be represented by a bi-infinite concatenation of B_n and $\overline{B_n}$. Since y can not contain a block in $\mathcal{F}$, there is an integer i such that $y_{[i+1,i+2^n]} = B_n$. Thus $x_{[-N,N]}$ occurs in $y_{[i+1,i+2^n]}$, and there is an integer $j \in [i+N+1, i+2^n-N]$ such that $x_{[-N,N]} = y_{[j-N,j+N]}$. $\square$

Lemma 6.3.6. *Suppose that φ is an automorphism of (M, σ_M). If there is a point $x \in M$ such that $\varphi(x) = x$, then $\varphi = id_M$.*

Proof. Suppose that $x \in M$ and $\varphi(x) = x$. There is a non-negative integer N such that for all $y, y' \in M$, $y_{[-N,N]} = y'_{[-N,N]}$ implies that $\varphi(y)_0 = \varphi(y')_0$.

Let $x \in X$. From Lemma 6.3.5, there is an integer j such that

$$y_{[-N,N]} = x_{[j-N,j+N]} = \sigma_M^j(x)_{[-N,N]}.$$

Then

$$\varphi(y)_0 = \varphi(\sigma_M^j(x))_0 = \sigma_M^j(\varphi(x))_0 = \sigma_M^j(x)_0 = x_j = y_0. \tag{6.11}$$

Since φ commutes with σ_M, (6.11) implies that $\varphi(x)_i = x_i$ for all $i \in \mathbb{Z}$. Since x is arbitrary, $\varphi = \mathrm{id}_M$. $\square$

In the following proof, for $x \in M$ and $i \leq j$, we write $x[i,j]$ insted of $x_{[i,j]}$.

Proof of Proposition 6.3.2. By Lemma 6.3.6, it is enough to show that whenever φ is an automorphism of (M, σ_M) and $x \in X$, then there are two numbers k, δ such that $k \in \mathbb{Z}$, $\delta \in \{0, 1\}$ and $\overline{\psi}^{\delta}(\sigma_M^k(\varphi(y))) = y$.

Suppose that φ is an automorphism of (M, σ_M) and $x \in M$. Let N be a non-negative integer such that whenever $y, y' \in M$ and $y[-N, N] = y'[-N, N]$, then $\varphi(y)_0 = \varphi(y')_0$. There are two integers n, m such that $2^n \geq 2N+1$, $0 \leq m \leq 2^n - 1$ and $m - 2^n + 1 \leq -N \leq N \leq m$. Then it follows that

$$y[0, 2^n - 1] = y'[0, 2^n - 1] \quad \Rightarrow \quad \varphi(y)_m = \varphi(y')_m \quad (y, y' \in M). \tag{6.12}$$

If we apply Lemma 6.3.3 to the point x and the number n, there is an integer p such that $x[p + i2^n, p + (i+1)2^n - 1] \in \{B_n, \overline{B_n}\}$ for all $i \in \mathbb{Z}$. We may assume that $p = 0$ since σ_M is one-to-one and $\overline{\psi}^{\delta} \sigma_M^k \varphi$ commutes with σ_M for any δ, k. Then

$$x = \cdots A_{-2} A_{-1} . A_0 A_1 A_2 \cdots = \langle A_i \rangle_{i \in \mathbb{Z}}$$

where $A_i = x[i2^n, (i+1)2^n - 1] \in \{B_n, \overline{B_n}\}$. If we put $\varphi(x) = y$, and apply Lemma 6.3.3 to the point y and the number n, then there is an integer q such that

$$y[q + i2^n, q + (i+1)2^n - 1] \in \{B_n, \overline{B_n}\}$$

for all $i \in \mathbb{Z}$. Since $|B_n| = |\overline{B_n}| = 2^n$, and since there are 2^n integers in $[m - 2^n + 1, m]$, there is a unique integer t such that $q + t2^n \in [m - 2^n + 1, m]$. If we put $k = q + t2^n$, then $k \in [m - 2^n + 1, m]$ and

$$y[k + i2^n, k + (i+1)2^n - 1] \in \{B_n, \overline{B_n}\}$$

for all $i \in \mathbb{Z}$. We write

$$y = \cdots C_{-2} C_{-1} C_0 C_1 C_2 \cdots = \langle C_i \rangle_{i \in \mathbb{Z}}$$

where $C_i = y[k + i2^n, k + (i+1)2^n - 1] \in \{B_n, \overline{B_n}\}$. We denote y_{m+i2^n} by d_i for $i \in \mathbb{Z}$. It is evident that d_i occurs in C_i. By (6.12), if $A_i = A_j$ then $d_i = d_j$; hence

$$A_i = A_j \quad \Longrightarrow \quad C_i = C_j. \tag{6.13}$$

Suppose that $A_0 = C_0$. Since a block in $\mathcal{F}$ does not occur in x, neither $A_0 A_0 A_0$ nor $\overline{A_0}\,\overline{A_0}\,\overline{A_0}$ occur in x. Hence there are two infinite subsets I, J of $\mathbb{Z}$ such that $J = \mathbb{Z} \setminus I$ and

$$A_i = A_0 \quad \text{and} \quad A_j = \overline{A_0} \qquad (i \in I, j \in J).$$

If $i \in I$, then $C_i = C_0$ by (6.13), so that $C_i = C_0 = A_0 = A_i$. If $j \in J$, then $C_j = \overline{C_0}$ since $J \cap I = \emptyset$ and $C_j, C_0 \in \{B_n, \overline{B_n}\}$, so that $C_j = \overline{C_0} = \overline{A_0} = A_j$. Therefore $A_r = C_r$ for all $r \in \mathbb{Z}$, and

$$x = \sigma_M^k(y) = \sigma_M^k(\varphi(x)) = \overline{\psi}^0(\sigma_M^k(\varphi(x))).$$

Suppose that $A_0 \neq C_0$, then $A_0 = \overline{C_0}$. By the same method as in the above case, we obtain $A_r = \overline{C_r}$ for all $r \in \mathbb{Z}$. Thus

$$x_i = \overline{\sigma_M^k(y)_i} = \overline{\sigma_M^k(\varphi(x))_i} = \overline{\psi}(\sigma_M^k(\varphi(x)))_i$$

for all $i \in \mathbb{Z}$, so that $x = \overline{\psi}(\sigma_M^k(\varphi(x)))$. The proof is done. $\square$

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국문초록

본 논문에서는 소픽 이동공간을 일반화한 이동공간들과 대합사상들에 대하여 연구한다. 소픽 이동공간을 주기점들의 조밀성에 대한 관점에서 일반화하면 코디드 기호 역학계이고, 엔트로피에 대한 관점에서 일반화하면 준소픽 이동공간이다. 이 두 이동공간들의 교집합에 포함되는 것이 기약 소픽 이동공간들의 모임이다. 본 논문에서는 기약 소픽 이동공간에서 성립하는 성질들 중에서, 어떤 성질들이 코디드 기호 역학계와 준소픽 이동공간에서도 성립하는지를 보인다.

또한, 이동공간이 대합사상을 하나 가질 때, 그 역학계와 동형이 아닌 역학계가 얼마나 많이 존재할 수 있는지 연구한다. 우리는 무한 싱크로나이즈드 기호 역학계가 대합사상을 하나 가지면, 그 역학계와 동형이 아닌 역학계가 무수히 많이 존재함을 증명한다. 하지만 코디드 기호 역학계는 이 성질을 만족하지 않는다. 이를 보이기 위해 대합사상을 갖는 코디드 기호 역학계를 건설하는데, 이 코디드 기호 역학계는 동형이 아닌 역학계가 두 개 뿐이다.

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